1. [Nov. 1986 Course 110 # 2] A system has two components placed in series so that the system fails if either of the two components fails. The second component is twice as likely to fail as the first. If the two components operate independently, and if the probability that the entire system will fail is 0.28, then what is the probability that the first component will fail?

~~(A) 0.28/3~~
$\boxed{\text{(B) } 0.10}$
~~(C) 0.56/3~~
(D) 0.20
(E) $\sqrt{0.14}$

$p$ = prob first fails

$2p$ = prob 2nd fails

$.28 = P[\text{either one fails}]$

$.28 = 1 - P[0 \text{ fail}]$

$.28 = 1 - (1-p)(1-2p)$

$2p^2 - 3p + .28 = 0$

$(p - .1)(2p - 2.8) = 0$

so $\boxed{p = .1}$ or $\cancel{p = 1.4}$ $p \le 1$

2. [Nov. 1986 Course 110 # 4] A pair of dice is tossed 10 times in succession. What is the probability of observing no 7's and no 11's in any of the 10 tosses?

(A) $(28/36)^{10}$
(B) $(30/36)^{10}(34/36)^{10}$
(C) $[1-(6/36)(2/36)]^{10}$
(D) $1-(8/36)^{10}$
(E) $[1-(6/36)^{10}][1-(2/36)^{10}]$

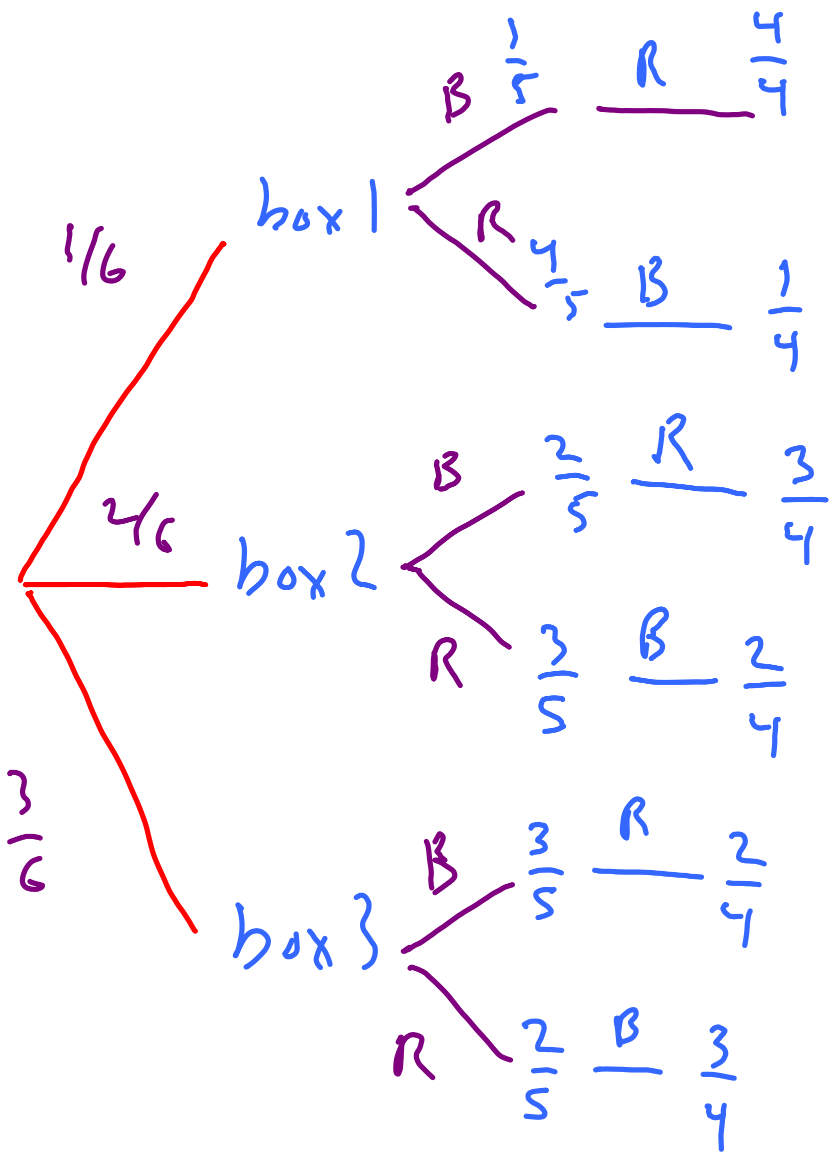
$$\left(P[\text{on one toss, no 7 or 11}]\right)^{10}$$

$$=\left(1-\frac{6+2}{36}\right)^{10}=\left(\frac{28}{36}\right)^{10}$$

3. [Nov. 1986 Course 110 # 7] Three boxes are numbered 1, 2, and 3. For $k = 1, 2$, or 3, box $k$ contains $k$ blue marbles and $5 - k$ red marbles. In a two-step experiment, a box is selected and 2 marbles are drawn from it without replacement. If the probability of selecting box $k$ is proportional to $k$, what is the probability that the 2 marbles drawn have different colors?

(A) 17/60
(B) 34/75
(C) 1/2
(D) 8/15
$\boxed{\text{(E) 17/30}}$

$$\left.\begin{aligned} P[\text{box } 1] &= c\cdot 1 \\ P[\text{box } 2] &= c\cdot 2 \\ P[\text{box } 3] &= c\cdot 3 \end{aligned}\right\} \text{total} = 1 \quad \text{so } c = \frac{1}{6}$$




$$\frac{1}{6}\cdot\frac{1}{5}\cdot\frac{4}{4}\cdot 2$$

$$2\cdot\frac{2}{6}\cdot\frac{2}{5}\cdot\frac{3}{4}$$

$$2\cdot\frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}$$

4. [Nov. 1986 Course 110 # 12] Let $X$ be a Poisson random variable with mean $\lambda$. If $\mathrm{P}[X = 1 \mid X \leq 1] = 0.8$, what is the value of $\lambda$?

(A) 4
(B) $-\ln(0.2)$
(C) 0.8
(D) 0.25
(E) $-\ln(0.8)$

$$P[X=1 \mid X\le 1] = \frac{P[X=1, X\le 1]}{P[X\le 1]}$$

$$= \frac{P[X=1]}{P[X\le 1]} = \frac{\lambda \cancel{e^{-\lambda}}}{\cancel{e^{-\lambda}} + \lambda \cancel{e^{-\lambda}}}$$

$$= \frac{\lambda}{1+\lambda} = 0.8$$

$$\lambda = .8 + .8\lambda$$

$$\lambda = 4$$

5. [Nov. 1986 Course 110 # 13] The following table represents the relative frequency of accidents per day in a city:

| Accidents | 0 | 1 | 2 | 3 | 4 or more |
|---|---|---|---|---|---|
| Relative Frequency | .55 | .2 | .1 | .1 | .05 |

Which of the following statements are true?

(I) The mean and the modal number of accidents are equal.

(II) The mean and the median number of accidents are equal.

(III) The median and modal number of accidents are equal.

(A) I only
(B) II only
$\boxed{\text{(C) III only}}$
(D) I, II, and III
(E) The correct answer is not given here.

mode = 0

median = 0 b/c P[#acc = 0] > .5

mean = 0 · .55 + 1(.2) + 2(.1) + ...

> 0

6. [Nov. 1986 Course 110 # 16] A random sampel of size 6 is selected with replacement from an urn that contains 10 red, 5 white, and 5 blue balls. What is the probability that the sample contains 2 balls of each color?

(A) 1/1024
(B) 1/646
$\boxed{\text{(C) } 45/512}$
(D) 75/646
(E) 45/64

R R W W B B

Prob of that

$$= \frac{10}{20} \cdot \frac{10}{20} \cdot \frac{5}{20} \cdot \frac{5}{20} \cdot \frac{5}{20} \cdot \frac{5}{20}$$

$$= \left(\frac{1}{2}\right)^2 \left(\frac{1}{4}\right)^4 = \left(\frac{1}{2}\right)^{10}$$

= prob of one specific order.

Answer = # of orders $\times \left(\frac{1}{2}\right)^{10}$

$$= \binom{6}{2} \cdot \binom{4}{2} \cdot \left(\frac{1}{2}\right)^{10} = \frac{45}{512}$$

choose which 2 are red

choose which 2 remaining are white

7. [Nov. 1986 Course 110 # 21] A drawer contains 6 blue socks and 4 white socks. Two of the socks are chosen at random $\boxed{\text{without}}$ replacement. What is the probability that the 2 socks are of the same color?

(A) 1/3
$\boxed{\text{(B) 7/15}}$
(C) 13/25
(D) 11/15
(E) 19/25

2 blue | 2 white

$$\frac{\binom{6}{2} + \binom{4}{2}}{\binom{10}{2}}$$

— # of ways to draw 2 socks

$$= \frac{\frac{6\cdot 5}{\cancel{2}} + \frac{4\cdot 3}{\cancel{2}}}{\frac{10\cdot 9}{\cancel{2}}} = \frac{7}{15}$$

8. [Nov. 1986 Course 110 # 24] A certain game involves rolling a pair of dice and watching for "sevens" to occur. What is the probability that it takes exactly 10 rolls to observe 8 sevens?

(A) $35^2/36^9$
$\boxed{\text{(B) } 5^2/6^8}$
(C) $5^2/6^{10}$
(D) $35^2/36^{10}$
(E) $45(5^2)/6^{10}$

$\frac{1}{6}$

7

__ __ ..... __ __

first 9 rolls, 7 sevens
2 non-seven

roll 10

$$\binom{9}{7}\left(\frac{1}{6}\right)^7\left(\frac{5}{6}\right)^2$$

$$= \frac{9\cdot 8}{2}\cdot\left(\frac{1}{6}\right)^7\left(\frac{5}{6}\right)^2\cdot\frac{1}{6}$$

$$= \frac{36\cdot 5^2}{6^{10}} = \frac{5^2}{6^8}$$

9. [Based on Nov. 1986 Course 110 # 36] Let $X_1, X_2$, and $X_3$ be independent random variables with common distribution P[$X = -3$] = 0.4, P[$X = 6$] = 0.6. Let $\bar{X} = (X_1 + X_2 + X_3)\ 3$ be the sample mean of $X_1, X_2, X_3$. What is the distribution of $\bar{X}$?

| # of $X_i = -3$ | $\bar{X}$ |
|---|---|
| 3 | $-3$ |
| 2 | $\frac{-3-3+6}{3} = 0$ |
| 1 | $(-3+6+6)\frac{1}{3} = 3$ |
| 0 | 6 |

$P[\text{all } 3\ X_i = -3] = (.4)^3 = .064$

$P[2\ X_i = -3] = \binom{3}{2}(.4)^2(.6) = .288$

$P[1\ X_i = -3] = \binom{3}{1}(.4)^1(.6)^2 = .432$

$P[0\ X_i = -3] = (.6)^3 = .216$

10. [Nov. 1986 Course 110 # 43] Let $S$ and $T$ be independent events, $P[S \cap T] = 1/10$, and $P[S \cap T'] = 1/5]$. What is $P[(S \cup T)']$?

(A) 3/10
(B) 11/30
(C) 7/15
(D) 8/15
(E) 9/10

$P[S \cap T] = \frac{1}{10}$ $\quad P[S \cap T'] = \frac{1}{5}$

so $P[S] = \frac{1}{10} + \frac{1}{5} = .3$

$P[S \cap T] = P[S]\,P[T]$

$\frac{1}{10} = \frac{3}{10} \cdot P[T]$, $\quad P[T] = \frac{1}{3}$

$P[(S \cup T)'] = 1 - P[S \cup T]$

$= 1 - \left( P(S) + P(T) - P(S \cap T) \right)$

$= 1 - \left( \frac{3}{10} + \frac{1}{3} - \frac{1}{10} \right)$

$= \frac{7}{15}$

11. [Based on May 1988 Course 110 # 1] One red die and one blue die are rolled. What is the probability that either the red die is a 1, 2, or 3, or the sum of the two dice is 11?

$A = \{\text{red die is } 1, 2, \text{ or } 3\}$

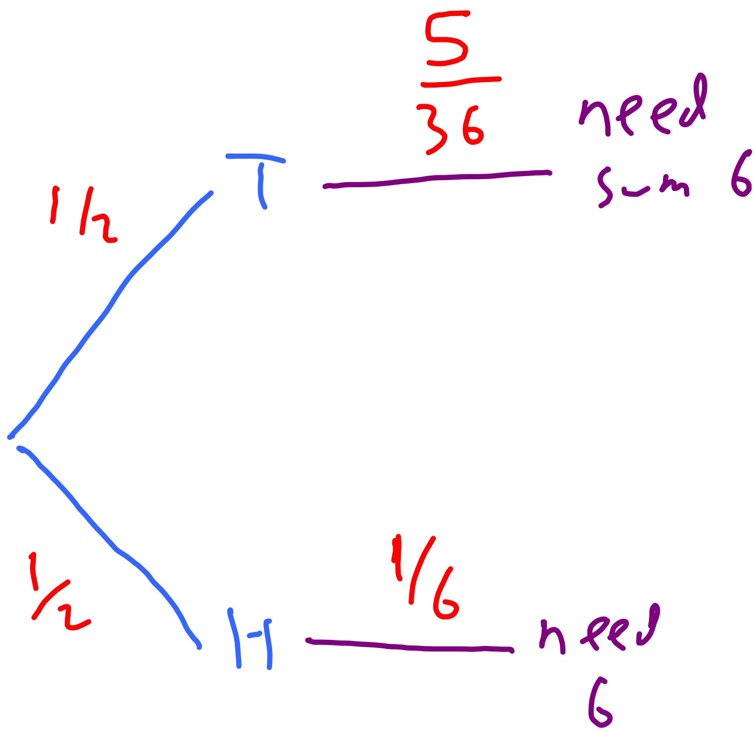
$B = \{\text{sum} = 11\}$ – need a 5 & a 6

$$P[A \cup B] = P[A] + P[B] - P[AB]$$

$$= \frac{3}{6} + \frac{2}{36} - \frac{0}{36}$$

$$= \frac{20}{36} = \frac{5}{9}$$

12. [Based on May 1988 Course 110 # 3] Suppose that I flip a coin. If it comes up heads, I then roll one die, and if it comes up tails, I roll two dice. What is the probability that the sum of the dice (or die) is a 6?




$$\frac{1}{2}\cdot\frac{5}{36} + \frac{1}{2}\cdot\frac{6}{36} = \boxed{\frac{11}{72}}$$

13. [Based on May 1988 Course 110 # 7] A bag contains 10 balls, 5 of which are white, 3 are red, and two are black. If three balls are drawn with replacement from the bag, what is the probability that one ball of each color is drawn?

$\underline{W}\ \underline{R}\ \underline{B}$ $\qquad \frac{5}{10} \cdot \frac{3}{10} \cdot \frac{2}{10} = \frac{30}{10^3}$

W B R $\qquad \frac{5}{10} \cdot \frac{2}{10} \cdot \frac{3}{10} = \frac{30}{10^3}$

$\vdots$

$3! = 6$ possible orders

Total prob $= 6 \cdot \frac{30}{10^3}$

$= 0.18$

14. [Based on May 1988 Course 110 # 8] Three fair dice are rolled. What is the probability that at least one of them is a 3?

$$1 - P[\text{none are } 3]$$

$$= 1 - \left(P[\text{first die} \neq 3]\right)^3$$

$$= 1 - \left(\frac{5}{6}\right)^3$$

15. [Based on May 1988 Course 110 # 16] Five cards are dealt from a standard deck without replacement. What is the probability that the hand results in exactly two kings, one of which is the king of spades, and exactly two queens.

2nd King  2 Queens

K of Sp

1 non-kings, non-queen

$$\frac{1\cdot\binom{3}{1}\cdot\binom{4}{2}}{\binom{52}{5}}\cdot\binom{44}{1}$$

16. [Based on May 1988 Course 110 # 22] A fair six-sided die is rolled until it comes up 3 or higher. What is the probability that total number of rolls made is even?

$N$ = total # of rolls

$N \sim \text{geometric}\left(p = \frac{4}{6}\right)$

$P[N=1] = \frac{4}{6} = \frac{2}{3}$

$P[N=2] = \left(\frac{1}{3}\right) \cdot \frac{2}{3}$

$P[N=3] = \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)$

$\vdots$

$P[N=k] = \left(\frac{1}{3}\right)^{k-1} \left(\frac{2}{3}\right)$

$P[N \text{ is even}] = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right) + \ldots$

$= \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left[1 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^4 + \ldots\right]$

$= \dfrac{\frac{1}{3} \cdot \frac{2}{3}}{1 - \left(\frac{1}{3}\right)^2} = \dfrac{\frac{2}{9}}{\frac{8}{9}} = \boxed{\frac{1}{4}}$

17. [Based on May 1988 Course 110 # 35] A jewelry collection consists of 25 fake diamonds and 10 real diamonds. If diamonds are drawn one at a time without replacement, what is the probability that exactly two fake diamonds are drawn before the second real diamond is drawn?

# of ways that work:

Real, Fake, Fake, Real

F, R, F, R

F, F, R, R

Prob of case 1: $\frac{10}{35} \cdot \frac{25}{34} \cdot \frac{24}{33} \cdot \frac{9}{32}$

Case 2: $\frac{25}{35} \cdot \frac{10}{34} \cdot \frac{24}{33} \cdot \frac{9}{32}$ = case 1

Total Prob: $3 \cdot \frac{10 \cdot 9 \cdot 25 \cdot 24}{35 \cdot 34 \cdot 33 \cdot 32}$

$= \frac{675}{5236}$

18. [Based on May 1990 Course 110 # 1] Suppose that $R$, $S$, and $T$ are independent events with $\mathrm{P}[R] = \mathrm{P}[S] = \mathrm{P}[T] = 1/3$. What is the probability that at least one of $R$, $S$, or $T$ occurs?

$$1 - P[\text{none of } R, S, \text{or } T \text{ occurs}]$$

$$= 1 - \left(P[\text{not } R]\right)\left(P[\text{not } S]\right)\left(P[\text{not } T]\right)$$

$$= 1 - \left(\frac{2}{3}\right)^3$$

$$= \boxed{\frac{19}{27}}$$

19. [Based on May 1990 Course 110 # 28] I have two unfair coins: the gold coin is biased so that it comes up heads with probability 0.3, and the silver coin comes up heads with probability 0.1. If I flip the gold coin 5 times and the silver coin 10 times, what is the probability that the gold coin comes up heads 3 times and the silver coin comes up heads 6 times?

$$P[\text{Gold is H 3 times}] \times P[\text{silver} = \text{H 6 times}]$$

$$= \binom{5}{3}(.3)^3(.7)^2\binom{10}{6}(.1)^6(.9)^4$$

20. [Based on May 1990 Course 110 # 42] Suppose that $X$, $Y$ and $Z$ are independent Poisson random variables with means 3, 1 and 4 respectively. Let $W = X + Y + Z$. Find $\mathrm{P}[W \leq 1]$.

Sum of independent Poissons is Poisson

so $W \sim Po(3+1+4) = Po(8)$

$$P[W \leq 1] = e^{-8} + 8e^{-8}$$

$$= \boxed{9e^{-8}}$$

21. [Based on May 1990 Course 110 # 50] A coin is biased so that the probability of tails is twice the probability of heads. If the coin is repeatedly flipped, what is the probability that the third head occurs on the fifth flip?

$$P[H] + P[T] = 1$$

$$P[H] + 2P[H] = 1,$$

so $P[H] = \frac{1}{3}$

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First 4 flips are 2 heads, 2 tails,

$5^{th}$ is head

$$\text{Prob} = \binom{4}{2} \cdot \left(\frac{1}{3}\right)^2 \cdot \left(\frac{2}{3}\right)^2 \cdot \frac{1}{3}$$

$$= 6 \cdot \frac{4}{3^5} = \frac{8}{3^4} = \boxed{\frac{8}{81}}$$