A.1 The “Laws” of Probability - Outline

Basic Principles of Probability

1. $0 \leq \mathrm{P}[A] \leq 1$
2. $\mathrm{P}[S] = 1 \quad \mathrm{P}[\emptyset] = 0$
3. If $A_1 \cap A_2 = \emptyset$, then $\mathrm{P}[A_1 \cup A_2] = \mathrm{P}[A_1] + \mathrm{P}[A_2]$

Example 1

Roll a fair six sided die

$S =$ “Sample Space” $=$ all possible outcomes $= \{1, 2, 3, 4, 5, 6\}$

$A = \{3, 4, 5, 6\} \qquad \mathrm{P}[A] = \dfrac{4}{6}$

$A_1 = \{3, 4, 5\} \qquad A_2 = \{6\} \qquad A_1 \cup A_2 = A$

$A_1 \cap A_2 = \emptyset$

$\mathrm{P}[A_1] = \dfrac{3}{6} \qquad \mathrm{P}[A_2] = \dfrac{1}{6} \qquad \mathrm{P}[A_1] + \mathrm{P}[A_2] = \mathrm{P}[A]$

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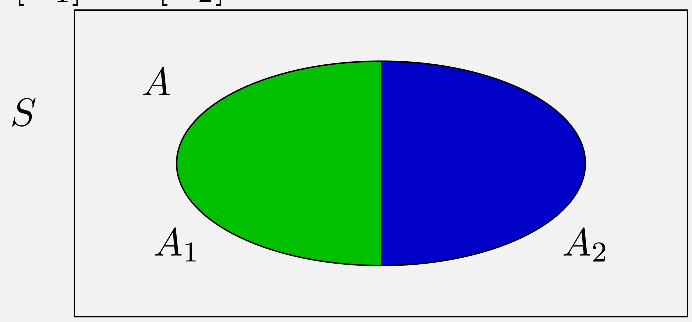
Example 2

Pick a point "at random" from S, where total area of $S = 1$

$\mathrm{P}[A] =$ area of $A \qquad A_1 \cap A_2 = \emptyset,\ A_1 \cup A_2 = A$

Area of A = area of A_1 + area of A_2

$\mathrm{P}[A] = \mathrm{P}[A_1] + \mathrm{P}[A_2]$



Unions and Intersections

1. $0 \le \mathrm{P}[A] \le 1$
2. $\mathrm{P}[S] = 1 \quad \mathrm{P}[\emptyset] = 0$
3. $\mathrm{P}[A \cup B] = \mathrm{P}[A] + \mathrm{P}[B] - \mathrm{P}[A \cap B]$

Example 1

Roll a fair six sided die

S = "Sample Space" = all possible outcomes = $\{1, 2, 3, 4, 5, 6\}$

A = "roll an odd number" $= \{1, 3, 5\} \qquad \mathrm{P}[A] = \dfrac{3}{6}$

B = "roll a 3 or less" $= \{1, 2, 3\} \qquad \mathrm{P}[B] = \dfrac{3}{6}$

$A \cup B = \{1, 2, 3, 5\} \qquad A \cap B = \{1, 3\}$

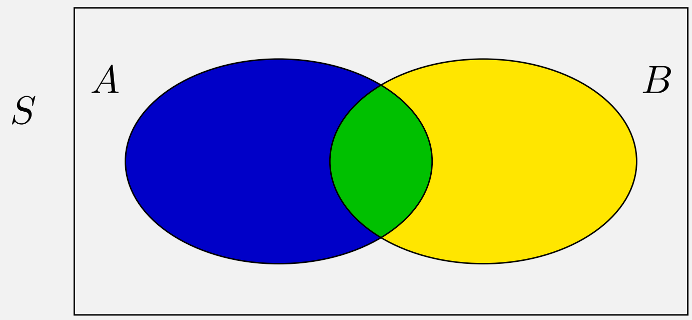
$$\mathrm{P}[A \cup B] = \frac{4}{6} = \frac{3}{6} + \frac{3}{6} - \frac{2}{6} = \mathrm{P}[A] + \mathrm{P}[B] - \mathrm{P}[A \cap B]$$

Unions and Intersections

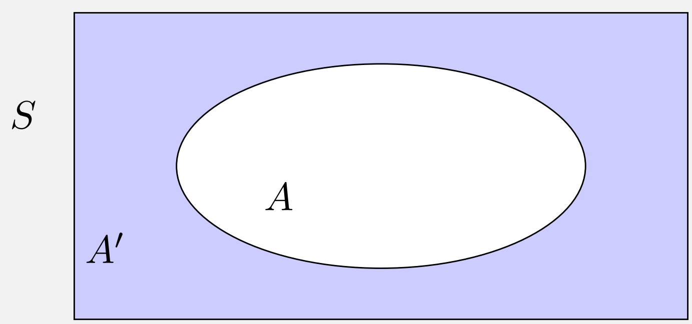
1. $0 \leq \mathrm{P}[A] \leq 1$
2. $\mathrm{P}[S] = 1 \quad \mathrm{P}[\emptyset] = 0$
3. $\mathrm{P}[A \cup B] = \mathrm{P}[A] + \mathrm{P}[B] - \mathrm{P}[A \cap B]$

Example 2

Pick a point "at random" from S, where total area of $S = 1$
$\mathrm{P}[A] =$ area of A.
Area of $A \cup B$ = area of A + area of B - area of $A \cap B$
$\mathrm{P}[A \cup B] = \mathrm{P}[A] + \mathrm{P}[B] - \mathrm{P}[A \cap B]$



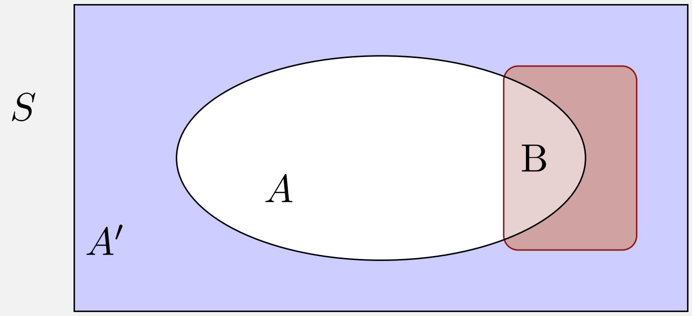
Complements



$A^c = A' =$ everything in S but not in A.
$A \cap A' = \emptyset \qquad A \cup A' = S$

E.g., if we roll a fair six sided die and $A =$ roll a 3 or higher, then $S = \{1, 2, 3, 4, 5, 6\}, \quad A = \{3, 4, 5, 6\}$ and $A' = \{1, 2\}$

Complements



$$A \cap A' = \emptyset \quad A \cup A' = S$$

$$\mathrm{P}[A] + \mathrm{P}[A'] = \mathrm{P}[A \cup A'] = \mathrm{P}[S] = 1$$

$$\mathrm{P}[A'] = 1 - \mathrm{P}[A]$$

Also, $(A')' = A$ so $\mathrm{P}[A] = 1 - \mathrm{P}[A']$
$B = (A \cap B) \cup (A' \cap B)$ and $(A \cap B) \cap (A' \cap B) = \emptyset$
so $\mathrm{P}[B] = \mathrm{P}[A \cap B] + \mathrm{P}[A' \cap B]$

Variation on SOA #2; S00.01

The probability that a visit to a primary care physician's (PCP) office results in either lab work or referral to a specialist is 60%. Of those coming to a PCP's office, 35% are referred to specialists and 50% require lab work. Determine the probability that a visit to a PCP's office results in both lab work and referral to a specialist.

Let L denote the event that the trip requires lab work, and S the event that it results in a referral to a specialist.
We are given:

$$\mathrm{P}[L \cup S] = 0.60$$

$$\mathrm{P}[S] = 0.35 \qquad \mathrm{P}[L] = 0.50$$

and we want $\mathrm{P}[L \cap S]$

$$\mathrm{P}[L \cup S] = \mathrm{P}[S] + \mathrm{P}[L] - \mathrm{P}[L \cap S]$$

$$0.60 = 0.35 + 0.50 - \mathrm{P}[L \cap S]$$

$$\mathrm{P}[L \cap S] = \boxed{0.25}$$

Variation on SOA #8; F01.09

Among a large group of patients recovering from shoulder injuries, it is found that 25% visit both a physical therapist and a chiropractor, and 85% visit at least one of these. The probability that a patient visits a chiropractor exceeds by 0.20 the probability that a patient visits a physical therapist. Determine the probability that a randomly chosen member of this group visits a physical therapist.

Let T denote the event that a patient visits a physical therapist, and C the event that a patient visits a chiropractor.

We are given:

$$\mathrm{P}[T \cap C] = 0.25 \qquad \mathrm{P}[T \cup C] = 0.85$$

$$\begin{aligned}
\mathrm{P}[C] &= 0.20 + \mathrm{P}[T] \\
\mathrm{P}[T \cup C] &= \mathrm{P}[C] + \mathrm{P}[T] - \mathrm{P}[T \cap C] \\
0.85 &= 0.20 + \mathrm{P}[T] + \mathrm{P}[T] - 0.25 \\
\mathrm{P}[T] &= \boxed{0.45}
\end{aligned}$$

A.1 The "Laws" of Probability - Outline

Formulas from last time

S = sample space, $\mathrm{P}[S] = 1$, $\mathrm{P}[\emptyset] = 0$

$\mathrm{P}[A \cup B] = \mathrm{P}[A] + \mathrm{P}[B] - \mathrm{P}[A \cap B]$

Special case: if $A_1 \cap A_2 = \emptyset$ then $\mathrm{P}[A_1 \cap A_2] = 0$
so $\mathrm{P}[A_1 \cup A_2] = \mathrm{P}[A_1] + \mathrm{P}[A_2] - 0 = \mathrm{P}[A_1] + \mathrm{P}[A_2]$
Generalization: if $A_i \cap A_j = \emptyset$ for all $i \neq j$ then

$$\mathrm{P}\left[\bigcup_{i=1}^{k} A_i\right] = \sum_{i=1}^{k} \mathrm{P}[A_i]$$

Complements: $A \cap A' = \emptyset$, $A \cup A' = S$

so $1 = \mathrm{P}[S] = \mathrm{P}[A] + \mathrm{P}[A']$
$\mathrm{P}[A] = 1 - \mathrm{P}[A']$ $\quad$ $\mathrm{P}[A'] = 1 - \mathrm{P}[A]$

Venn Diagrams

Venn diagrams are a visual way to represent unions and intersections.

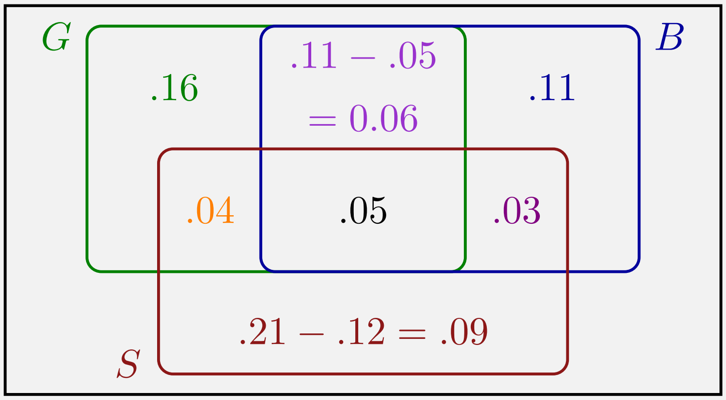
They allow us to easily deal with more complicated cases than what we have seen so far.

Variation on SOA #1; S.03.01

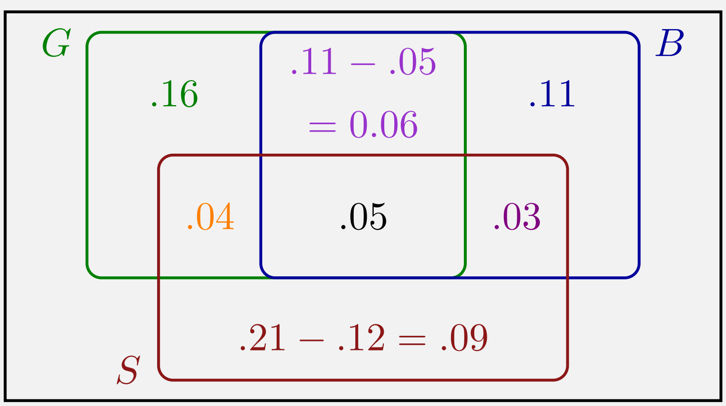
A survey of a group's viewing habits found the following:

1. 31% watched gymnastics, 25% watched baseball, and 21% watched soccer
2. 11% watched gymnastics and baseball, 8% watched baseball and soccer, and 9% watched gymnastics and soccer
3. 5% watched all three sports.

Find the percentage of the group that watched none of the three sports.



Variation on SOA #1; S03.01



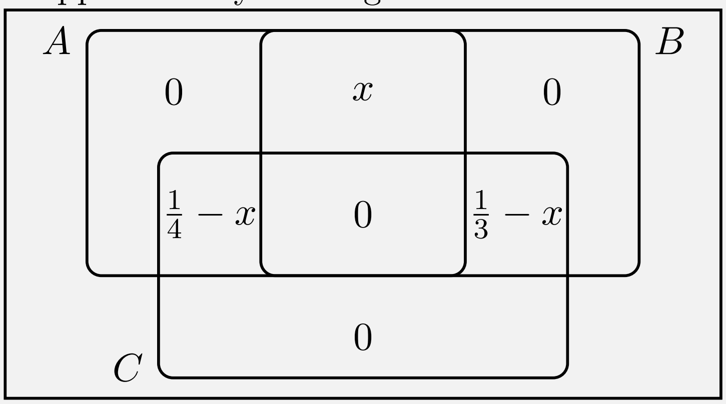
We want the probability that someone watches none of these sports, so we want

$$1 - 0.16 - 0.06 - 0.11 - 0.04 - 0.05 - 0.03 - 0.09 = \boxed{0.46}$$

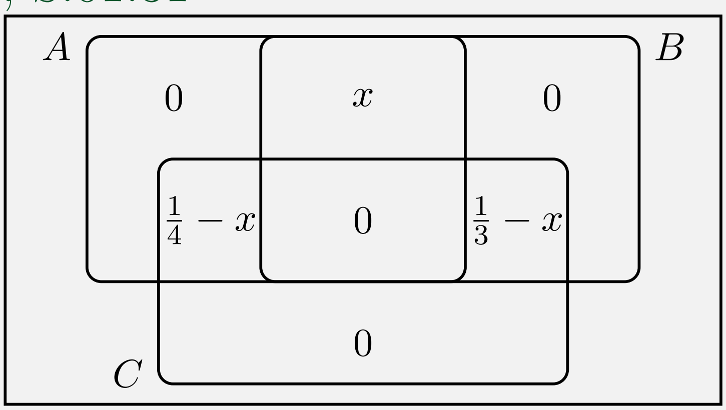
SOA #15; S.01.31

An insurer offers a health plan to the employees of a large company. The individual employees may choose exactly two of the supplementary coverages A, B, and C, or they may choose no supplementary coverage. The proportions of the company's employees that choose coverages A, B, and C are 1/4, 1/3, and 5/12, respectively.

Determine the probability that a randomly chosen employee will choose no supplementary coverage.



SOA #15; S.01.31



$$\frac{5}{12} = \left(\frac{1}{4} - x\right) + \left(\frac{1}{3} - x\right) = \frac{7}{12} - 2x, \quad x = \frac{1}{12}$$

$$\begin{aligned} \mathrm{P}[(A \cup B \cup C)'] &= 1 - x - \left(\frac{1}{4} - x\right) - \left(\frac{1}{3} - x\right) \\ &= 1 - \frac{1}{12} - \frac{3-1}{12} - \frac{4-1}{12} = \frac{1}{2} \end{aligned}$$

SOA #15; S.01.31

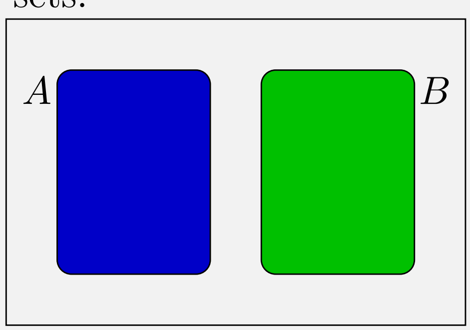
Alternatively, $\mathrm{P}[A] + \mathrm{P}[B] + \mathrm{P}[C]$ counts each person who buys coverage twice since each person buys exactly 2 coverages. $\mathrm{P}[A \cup B \cup C]$ counts each person who buys coverage once, so

$$\begin{aligned} \mathrm{P}[A] + \mathrm{P}[B] + \mathrm{P}[C] &= 2 \cdot \mathrm{P}[A \cup B \cup C] \\ \frac{1}{4} + \frac{1}{3} + \frac{5}{12} &= 2 \cdot \mathrm{P}[A \cup B \cup C] \\ \frac{1}{2} &= \mathrm{P}[A \cup B \cup C] \\ \mathrm{P}[(A \cup B \cup C)'] &= 1 - \frac{1}{2} = \boxed{\frac{1}{2}} \end{aligned}$$

Mutually Exclusive Events

A and B are said to be mutually exclusive (or mutually disjoint) events if $A \cap B = AB = \emptyset$. Note that this means that $\mathrm{P}[A \cap B] = \mathrm{P}[AB] = 0$.

From the point of view of a Venn diagram, we can draw them as non-overlapping sets.



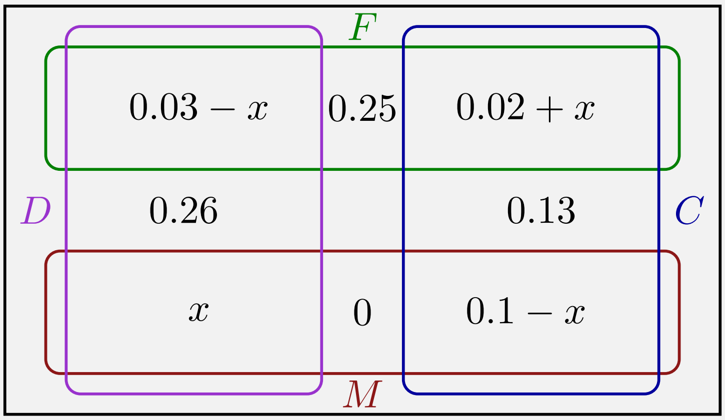
If we have 4 or more events, typically some of them will be mutually exclusive to simplify any Venn diagrams we need to draw.

Example

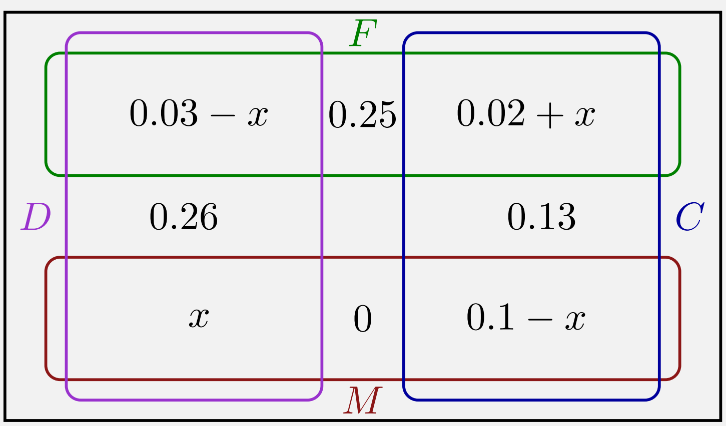
A survey of a group's viewing habits found the following:

1. 30% watched Fox News, 25% watched CNN, 10% watched MSNBC and 29% watched the Daily Show
2. No one watched both Fox and MSNBC, and no one watched both CNN and the Daily Show.
3. 13% watched only CNN, 26% watched only the Daily Show, but no one watched only MSNBC.

Find the percentage of the group that watched none of the four.



Example



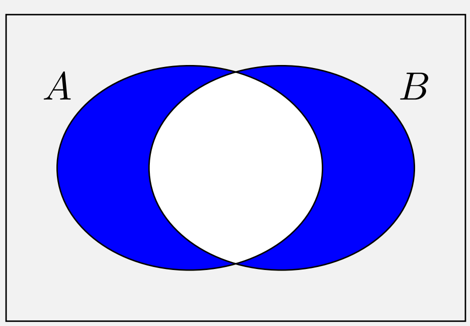
We want the probability that someone watches none of these 4, which is

$$
\begin{aligned}
&1 - (0.03 - x + 0.25 + 0.02 + x) - 0.26 - 0.13 - (x + 0.1 - x) \\
&= 1 - 0.30 - 0.26 - 0.13 - 0.1 \\
&= \boxed{0.21}
\end{aligned}
$$

A.1 The “Laws” of Probability - Outline

Key Words

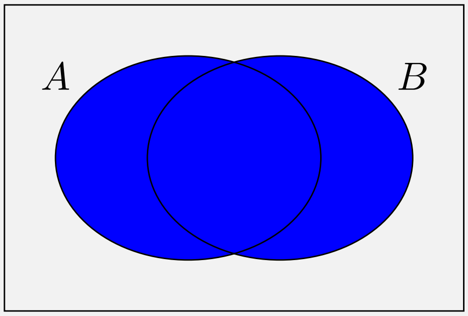
For us, A or B means union ($A \cup B$), i.e., $A \cup B$ means A or B or both.
A and B means intersection ($A \cap B$, also denoted as AB)
A or B but not both means we want to start with $A \cup B$ but exclude the intersection.



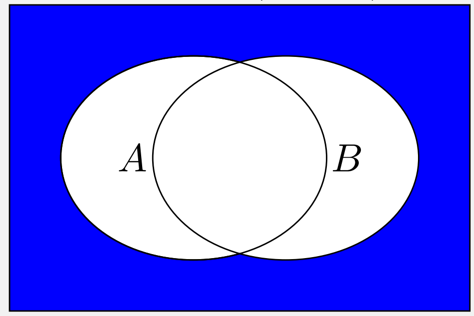
Algebraically, that means we want $(A \cap B') \cup (A' \cap B)$
What about neither A nor B?

De Morgan's Laws

$A \cup B$



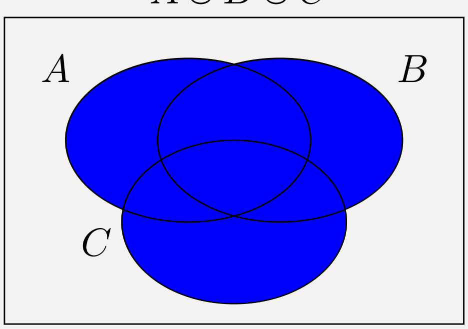
$A' \cap B' = (A \cup B)'$



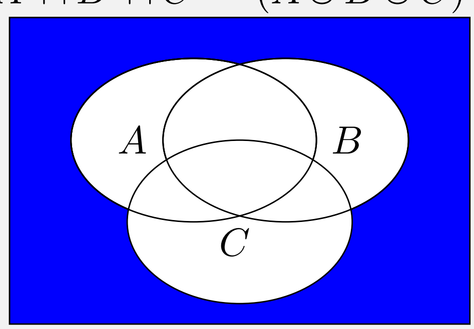
The left hand diagram is A or B, while the right hand diagram is neither A nor B.

De Morgan's Laws

$A \cup B \cup C$



$A' \cap B' \cap C' = (A \cup B \cup C)'$



De Morgan's Laws

General version of De Morgan's Laws:

$$\left[\bigcup_{i=1}^{k} A_i\right]' = \left[\bigcap_{i=1}^{k} A_i'\right]$$
$$\left[\bigcap_{i=1}^{k} A_i\right]' = \left[\bigcup_{i=1}^{k} A_i'\right]$$

Example

An auto insurance company has 10,000 policyholders. Each policyholder is classified as

1. young or old; and
2. male or female;

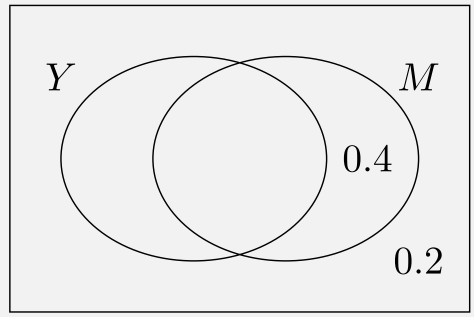
Of these policyholders, 2,000 are neither young nor male, and 6,000 are either young or female.
How many policyholders are young?

Let Y denote young policyholders, and M male policyholders.
Neither young nor male is $(Y \cup M)' = Y' \cap M'$.
We are given $\mathrm{P}[(Y \cup M)'] = \mathrm{P}[Y' \cap M'] = 0.2$ and $\mathrm{P}[Y \cup M'] = 0.6$.

We want $\mathrm{P}[Y]$.

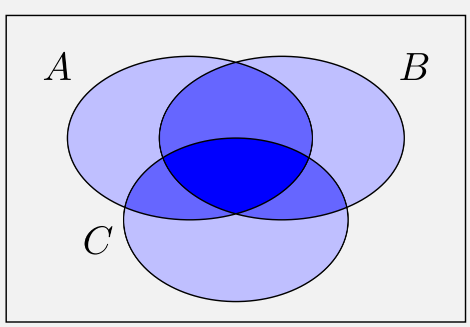
Example

$$
\begin{aligned}
P[Y' \cap M'] &= 0.2 \\
P[Y \cup M'] &= 0.6 \\
P[(Y \cup M')'] &= 1 - 0.6 \\
P[Y' \cap M] &= 0.4 \\
P[Y'] &= P[Y' \cap M'] + P[Y' \cap M] \\
P[Y'] &= 0.2 + 0.4 = 0.6 \\
P[Y] &= 1 - 0.6 = \boxed{0.4}
\end{aligned}
$$



Inclusion-Exclusion

$$
\begin{aligned}
P[A \cup B] &= P[A] + P[B] - P[AB] \\
P[A \cup B \cup C] &= P[A] + P[B] + P[C] \\
&- P[AB] - P[AC] - P[BC] \\
&+ P[ABC]
\end{aligned}
$$



Inclusion-Exclusion

$$
\begin{aligned}
\mathrm{P}[A \cup B \cup C \cup D] = {} & \mathrm{P}[A] + \mathrm{P}[B] + \mathrm{P}[C] + \mathrm{P}[D] \\
& - \mathrm{P}[AB] - \mathrm{P}[AC] - P[AD] - \mathrm{P}[BC] - \mathrm{P}[BD] - \mathrm{P}[CD] \\
& + \mathrm{P}[ABC] + P[ABD] + \mathrm{P}[ACD] + \mathrm{P}[BCD] \\
& - \mathrm{P}[ABCD]
\end{aligned}
$$

Variation on SOA #1; S.03.01 revisited

A survey of a group's viewing habits found the following:

1. 31% watched gymnastics, 25% watched baseball, and 21% watched soccer
2. 11% watched gymnastics and baseball, 8% watched baseball and soccer, and 9% watched gymnastics and soccer
3. 5% watched all three sports.

Find the percentage of the group that watched none of the three sports.

Again, let G, B and S denote those who watch gymnastics, baseball, and soccer respectively. Then our inclusion-exclusion formula gives us

$$
\begin{aligned}
\mathrm{P}[G \cup B \cup S] &= \mathrm{P}[G] + \mathrm{P}[B] + \mathrm{P}[S] - \mathrm{P}[GB] - \mathrm{P}[BS] - \mathrm{P}[GS] + \mathrm{P}[GBS] \\
&= 0.31 + 0.25 + 0.21 - 0.11 - 0.08 - 0.09 + 0.05 \\
&= 0.54
\end{aligned}
$$

$$\mathrm{P}[(G \cup B \cup S)'] = 1 - 0.54 = \boxed{0.46}$$