A1. Measurement of Interest

A.1.1 Interest Accumulation – Part 1

What is Interest?

Two Perspectives

Amount Function, $A(t)$

Accumulation Function, $a(t)$

Exercises

What is Interest?

Today you open a savings account at your local bank with an initial deposit of 100. One year later, you check the savings account balance and find it to be 105.

Why did your account balance grow? INTEREST!

The initial deposit of 100 is called the **principal**.

The account balance of 105 is called the **accumulated value (AV)** at time 1, where time is measured in years.

The difference between the accumulated value and the principal is called the amount of interest, or just **interest**.

$$\text{interest} = 105 - 100 = 5$$

Interest is the compensation a borrower of capital pays to the lender for its use. In other words, interest is the rent for borrowing the principal.

Two Perspectives

Every financial transaction can be viewed from two different perspectives:

1. lender of the capital
2. borrower of the capital

Savings Account

- depositor is the lender
- bank is the borrower

Car Loan

- bank is the lender
- car buyer is the borrower

Amount Function, $A(t)$

Assume that given the original principal invested, the accumulated value at any point in time can be determined.

Let $A(t)$ represent the accumulated value at time t for an original investment of k.

Properties of $A(t)$

1. $A(0) = k$
2. $A(t)$ is generally increasing.
3. If interest accrues continuously the function will be continuous.

accrue – to increase in value or amount as time passes

The amount of interest earned for period n is

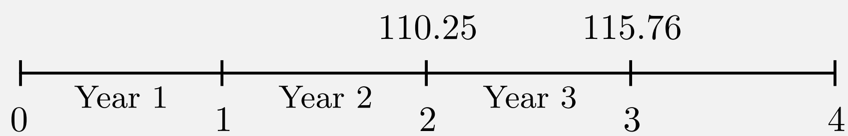
$$I_n = A(n) - A(n-1)$$

Amount Function, $A(t)$ cont.

An investment of 100 is made into a fund at time $t = 0$. The fund develops the following balances over the next 4 years:

t	$A(t)$
0	100.00
1	105.00
2	110.25
3	115.76
4	121.55

Find the amount of interest earned during the third year.



$$I_3 = 115.76 - 110.25 = \boxed{5.51}$$

Note $A(3)$ is the accumulated value *at time 3* and I_3 is the interest *for year 3*.

Accumulation Function, $a(t)$

The accumulation function is a special case of the amount function where the original investment is one unit (i.e. 1).

$a(t)$ is the AV at time t of an original investment of 1.

Recall that $A(t)$ is the accumulated value at time t of an original investment of k, thus

$$A(t) = k \cdot a(t)$$

The second and third properties of $A(t)$ also hold for $a(t)$.

As you will see in later lessons, $A(t)$ and $a(t)$ frequently can be used interchangeably.

Accumulation Function, $a(t)$ cont.

Given $A(t) = t^2 + 3t + 5$, find the corresponding $a(t)$.

First find k

$$\begin{aligned} A(0) &= k \\ 0^2 + 3(0) + 5 &= k \\ 5 &= k \end{aligned}$$

now using $A(t) = k \cdot a(t)$

$$\begin{aligned} A(t) &= 5 \cdot a(t) \\ a(t) &= \tfrac{1}{5} \cdot A(t) = \tfrac{1}{5}\left(t^2 + 3t + 5\right) \\ a(t) &= \tfrac{1}{5}t^2 + \tfrac{3}{5}t + 1 \\ a(0) &= 1 \quad \checkmark \end{aligned}$$

Exercise 1

You are given:

(i) $a(t) = 1 + 0.1t$

(ii) \$1000 is invested at time 0.

Determine the amount of interest earned during year 8.

$$\begin{aligned} A(t) &= 1000(1 + 0.1t) \\ I_8 &= A(8) - A(7) \\ &= 1000(1 + 0.1(8)) - 1000(1 + 0.1(7)) \\ &= 1000(0.1)(8 - 7) \\ &= \boxed{100} \end{aligned}$$

Find the amount of interest during year n.

$$\begin{aligned} I_n &= A(n) - A(n-1) \\ &= 1000(1 + 0.1(n)) - 1000(1 + 0.1(n-1)) \\ &= 1000(0.1)(n - (n-1)) \\ &= 100 \end{aligned}$$

Exercise 2

You are given:

(i) $a(t) = bt^2 + c$

(ii) \$100 invested at time 0 will be worth \$200 at time 10.

Find the accumulated value at time 15 of \$500 invested at time 0.

From bullet (ii):

$A(0) = 100 \Rightarrow A(0) = 100 \cdot a(0) \Rightarrow A(t) = 100 \cdot a(t)$ and
$A(10) = 200 \Rightarrow 100 \cdot a(10) = 200 \Rightarrow a(10) = 2.$

We know that $a(0) = 1$, and $a(10) = 2$:

$$\begin{aligned} a(0) &= 1 \\ b \cdot 0^2 + c &= 1 \\ c &= 1 \end{aligned} \qquad\qquad \begin{aligned} a(10) &= 2 \\ b \cdot 10^2 + 1 &= 2 \\ b &= 0.01 \end{aligned}$$

The accumulated value of \$500 in 15 years is

$$500 \cdot a(15) = 500 \left[0.01(15^2) + 1\right] = \boxed{1625}$$

A.1.2 Interest Accumulation – Part 2

The Effective Rate of Interest, i_n

Accumulation Factor, $1 + i_n$

Exercises

The Effective Rate of Interest, i_n

100 invested at time 0 is worth 110 at time 1.

$$\begin{array}{cc} 100 & \\ \vdash\!\!\!-\!\!\!-\!\!\!-\!\!\!- & \!\!\!\dashv \\ 0 & 1 \\ & 100 \\ & \underline{+10} \\ & 110 \end{array}$$

We can think of the 110 as a return of the 100 plus 10 of interest. What would you say the rate of interest is?

$$\frac{10}{100} = 10\% = i_1$$

The annual effective rate of interest for year 1 is 10%.

The Effective Rate of Interest, i_n cont.

The formal definition is:

The effective rate of interest, i, is the amount of money that one unit invested at the beginning of a period will earn during the period, where interest is paid at the end of the period.

Notes:

1. The term "effective" is used for rates of interest in which interest is paid once per measurement period (as opposed to "nominal" rates which we will discuss later).
2. The effective rate of interest is usually expressed as a percentage, e.g. 5%. 5% is equivalent to 0.05 per unit of principal.
3. Amount of principal remains constant throughout the period.

The Effective Rate of Interest, i_n cont.

The formal definition is:

The effective rate of interest, i, is the amount of money that one unit invested at the beginning of a period will earn during the period, where interest is paid at the end of the period.

We can rewrite the definition in terms of the amount function:

$$i = a(1) - a(0) = \frac{a(1) - a(0)}{a(0)} = \frac{k \cdot a(1) - k \cdot a(0)}{k \cdot a(0)}$$

$$= \frac{A(1) - A(0)}{A(0)} = \frac{I_1}{A(0)}$$

Thus an alternate definition is:

The effective rate of interest, i, is the amount of interest earned during the period divided by the amount of principal invested at the beginning of the period.

The Effective Rate of Interest, i_n cont.

The effective rate of interest, i, is the amount of interest earned during the period divided by the amount of principal invested at the beginning of the period.

The effective rate of interest does not have to be the same for each period.

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{I_n}{A(n-1)} \quad \text{for } n = 1, 2, 3, \ldots$$

We can also write the effective rate of interest in terms of the accumulation function

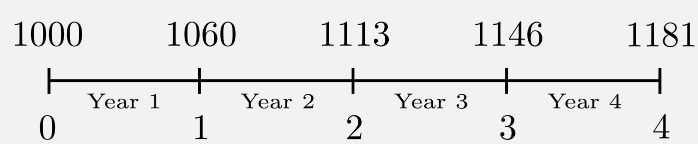
$$i_n = \frac{a(n) - a(n-1)}{a(n-1)}$$

The Effective Rate of Interest, i_n cont.

Example

An investment of 1000 is made into a fund at time $t = 0$. The fund develops the following balances over the next 4 years:

t	$A(t)$
0	1000
1	1060
2	1113
3	1146
4	1181

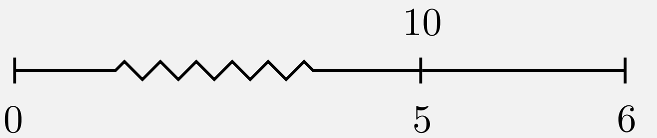


Find i_1 and i_3, the effective rates of interest for years one and three respectively.

$$i_1 = \frac{1060 - 1000}{1000} = 6\% \qquad i_3 = \frac{1146 - 1113}{1113} = 2.96\%$$

Accumulation Factor, $1 + i_n$

An investment is worth 10 at time 5. If $i_6 = 15\%$, how much is it worth at time 6?



$$\begin{array}{r} 10 \\ \underline{+15\%(10)} \\ 1.15(10) \end{array}$$

$$A(6) = A(5) + i_6 \cdot A(5) = (1 + i_6)A(5)$$

$1 + i_6$ is called the **accumulation factor** for year 6 because it is used to **accumulate** the investment from time 5 to time 6.

Accumulation Factor, $1 + i_n$ - cont.

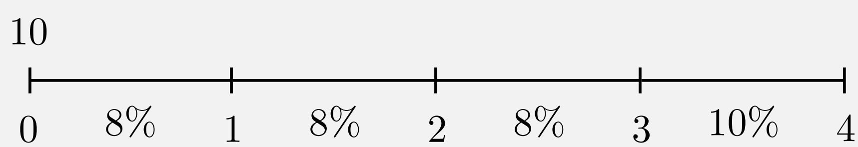
$1 + i_n$ - accumulation factor for period n

$$1 + i_n = 1 + \frac{a(n) - a(n-1)}{a(n-1)} = \frac{a(n)}{a(n-1)}$$

We can accumulate money from time $n - 1$ to time n by multiplying by $1 + i_n$ or $a(n)/a(n-1)$.

Example

You are given $i_1 = i_2 = i_3 = 8\%$ and $i_4 = 10\%$. Find the accumulated value of 10 in 4 years.



$$A(4) = 10(1.08)(1.08)(1.08)(1.10) = 10(1.08)^3(1.10) = \boxed{13.86}$$

Accumulation Factor, $1 + i_n$ - cont.

Example

An investment of 1000 is made into a fund at time $t = 0$. The fund develops the following balances over the next 4 years:

t	$A(t)$
0	1000
1	1060
2	1113
3	1146
4	1181

1000	1060	1113	1146	1181
0	1	2	3	4
	5000			X

If 5000 is invested in the same fund at time 1, find the accumulated value of this deposit at time 4.

$$X = 5000 \cdot \frac{a(4)}{a(1)} = 5000 \cdot \frac{A(4)}{A(1)} = 5000 \cdot \frac{1181}{1060} = \boxed{5570.75}$$

Alternatively: $\dfrac{1181}{1060} = \dfrac{X}{5000} \quad \Rightarrow \quad X = 5570.75$

Exercise 1

Given $A(t) = 10(1.08)^t$. Find i_3 and i_7.

i_3 is the effective rate of interest for year 3

$$i_3 = \frac{A(3) - A(2)}{A(2)} = \frac{10(1.08)^3 - 10(1.08)^2}{10(1.08)^2} = 1.08 - 1 = 0.08$$

i_7 is the effective rate of interest for year 7

$$i_7 = \frac{A(7) - A(6)}{A(6)} = \frac{10(1.08)^7 - 10(1.08)^6}{10(1.08)^6} = 1.08 - 1 = 0.08$$

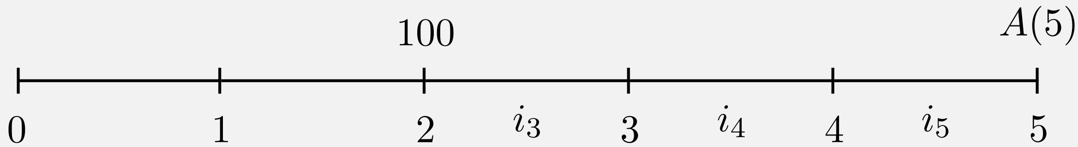
Hmm. What about year n?

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{10(1.08)^n - 10(1.08)^{n-1}}{10(1.08)^{n-1}} = 1.08 - 1 = 0.08$$

This is compound interest. It is the interest assumption for almost every FM problem.

Exercise 2

Given $A(2) = 100$ and $i_n = 0.03n$, determine $A(5)$.

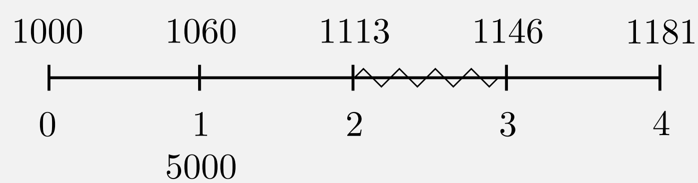


$$\begin{aligned} A(5) &= 100(1+i_3)(1+i_4)(1+i_5) \\ &= 100(1+0.03(3))(1+0.03(4))(1+0.03(5)) \\ &= \boxed{140.39} \end{aligned}$$

Exercise 3

An investment of 1000 is made into a fund at time $t = 0$. The fund develops the following balances over the next 4 years:

t	$A(t)$
0	1000
1	1060
2	1113
3	1146
4	1181



If 5000 is invested in the same fund at time 1, find the amount of interest earned on this deposit between time 2 and 3.

$$A^*(2) = 5000 \cdot \tfrac{1113}{1060} = 5250$$

$$A^*(3) = 5000 \cdot \tfrac{1146}{1060} = 5405.66$$

$$I_3^* = 5405.66 - 5250 = \boxed{155.66}$$

A.1.3 Present Value – Part 1

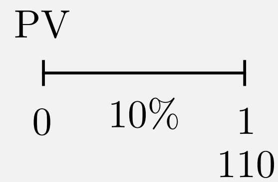
- What is the Present Value?
- Effective Rate of Discount, d_t
- Discount Factor, $1 - d_t$ or v_t
- Summary of Ways to Accumulate and Discount
- Exercises

What is the Present Value?

Suppose you need 110 one year from today. If you can invest money at an effective interest rate of 10%, then how much do you need to invest today to have exactly 110 one year from today?

The amount you need to invest is called the **present value** (PV).

In this case we know the AV and are solving for the PV. In the previous lesson we were given the PV and were solving for the AV.



$$PV(1.10) = 110$$

$$PV = \frac{110}{1.10} = \boxed{100}$$

What is the Present Value? cont.

Given an accumulation function $a(t)$, what is the present value of 1 at time t?

$$\begin{array}{ccc} \text{PV} & & \\ \vdash & \text{———————} & \dashv \\ 0 & & t \\ & & 1 \end{array}$$

$$\text{PV} \cdot a(t) = 1$$

$$\text{PV} = \frac{1}{a(t)}$$

To accumulate cash flows from time 0 to time t we multiply by $a(t)$ and to **discount** cash flows from time t to time 0 we divide by $a(t)$.

$1/a(t)$ is called the **discount function** and is sometimes denoted $a^{-1}(t)$.

Effective Rate of Discount, d_t

Recall the example from the previous lesson. 100 invested at time 0 is worth 110 at time 1.

$$\begin{array}{ccc} 100 & & \\ \vdash & \text{———} & \dashv \\ 0 & & 1 \\ & & 100 \\ & & \underline{+10} \\ & & 110 \end{array}$$

$$i_1 = \frac{10}{100} = \frac{1}{10} = 10\%$$

We can also view these cash flows as borrowing 110, but paying 10 in interest at time 0.

$$\begin{array}{ccc} 110 & & \\ \underline{-10} & & \\ 100 & & \\ \vdash & \text{———} & \dashv \\ 0 & & 1 \\ & & 110 \end{array}$$

The rate of interest paid at the beginning of the period is

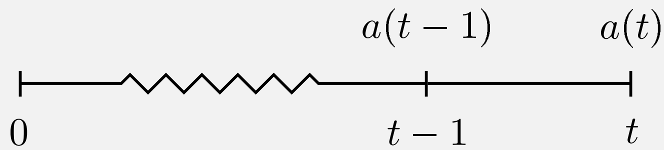
$$d_1 = \frac{10}{110} = \frac{1}{11} = 9.09\%$$

d_1 is called the effective rate of discount for year 1.

Effective Rate of Discount, d_t - cont.

Why do we need another way to measure interest? Mathematically we don't. It is for convenience.

The effective rate of discount is a measure of interest paid at the beginning of the period.



$$d_t = \frac{a(t) - a(t-1)}{a(t)}$$

$$i_t = \frac{a(t) - a(t-1)}{a(t-1)}$$

In words, d_t is the amount of interest earned in period t divided by the amount invested at the end of the period.

Discount Factor, $1 - d_t$ or v_t

Recall that $1 + i_t$ is the accumulation factor.

$1 - d_t$ is the **discount factor**. It discounts the value of a cash flow from time t to time $t - 1$.

In our example from slide 4, $d_1 = \dfrac{1}{11}$.

$$110(1 - \tfrac{1}{11}) = 100$$

Let's write the discount factor in terms of the accumulation function

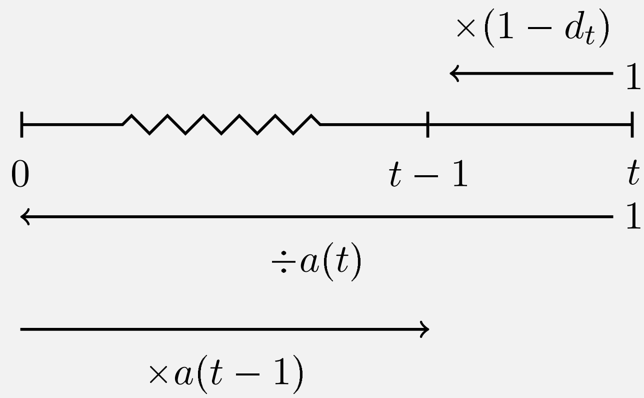
$$d_t = \frac{a(t) - a(t-1)}{a(t)}$$

$$1 - d_t = \frac{a(t)}{a(t)} - \frac{a(t) - a(t-1)}{a(t)}$$

$$1 - d_t = \frac{a(t-1)}{a(t)}$$

Discount Factor, $1 - d_t$ or v_t - cont.

$$1 - d_t = \frac{a(t-1)}{a(t)}$$

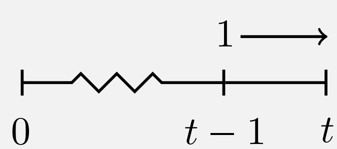


The discount factor is also written as v_t, that is

$$v_t \equiv 1 - d_t$$

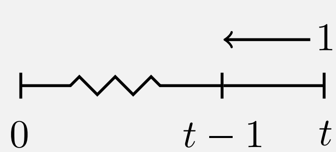
Summary of Ways to Accumulate and Discount

Accumulate 1 from time $t-1$ to t



1. $\times \dfrac{a(t)}{a(t-1)}$
2. $\times(1+i_t)$
3. $\div(1-d_t)$
4. $\div v_t$

Discount 1 from time t to $t-1$



1. $\times \dfrac{a(t-1)}{a(t)}$
2. $\div(1+i_t)$
3. $\times(1-d_t)$
4. $\times v_t$

Exercise 1

Given $a(t) = 1 + 0.08t$, find i_4 and d_4.

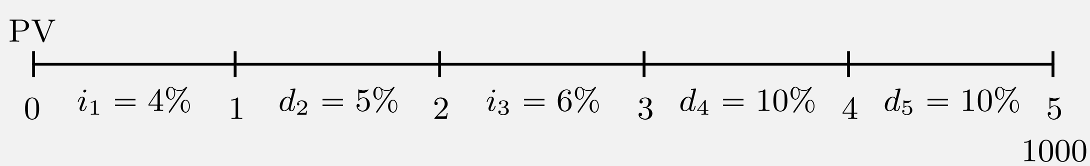
$$a(3) = 1 + 0.08(3) = 1.24$$
$$a(4) = 1 + 0.08(4) = 1.32$$
$$i_4 = \frac{1.32 - 1.24}{1.24} = 0.0645$$
$$d_4 = \frac{1.32 - 1.24}{1.32} = 0.0606$$

Exercise 2

John can earn the following rates for the next five years:

- Year 1: effective rate of interest = 4%
- Year 2: effective rate of discount = 5%
- Year 3: effective rate of interest = 6%
- Years 4-5: effective rate of discount = 10%

John needs to have 1000 in 5 years. How much should he invest now?



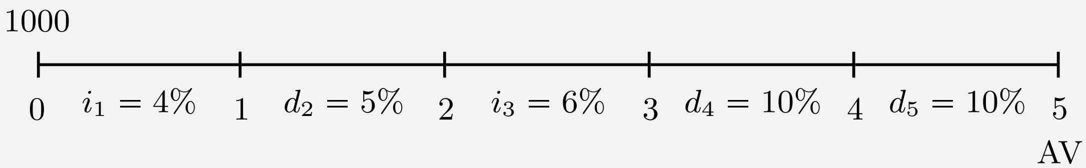
$$PV = 1000(1-0.1)(1-0.1)(1.06)^{-1}(1-0.05)(1.04)^{-1} = \boxed{698.02}$$

Exercise 3

John can earn the following rates for the next five years:

- Year 1: effective rate of interest = 4%
- Year 2: effective rate of discount = 5%
- Year 3: effective rate of interest = 6%
- Years 4-5: effective rate of discount = 10%

John invests 1000 today. How much will John have in 5 years?

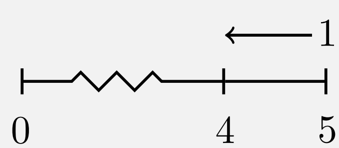


$$\text{AV} = 1000(1.04)(1-0.05)^{-1}(1.06)(1-0.1)^{-1}(1-0.1)^{-1} = \boxed{1432.62}$$

Exercise 4

Given $a(t) = 1.08^t$, find v_5.

v_5 is the discount factor for year 5.



$$v_5 = \frac{a(4)}{a(5)} = \frac{1.08^4}{1.08^5} = 1.08^{-1} = \boxed{0.92593}$$

A.1.4 Present Value – Part 2

Relationship between i_t and d_t

Terminology Note

Exercises

Relationship between i_t and d_t

Write i_t in terms of d_t. Hint: first write i_t in terms of $a(t)$.

$$\begin{aligned} i_t &= \frac{a(t) - a(t-1)}{a(t-1)} \\ &= \frac{a(t) - a(t-1)}{a(t)} \cdot \frac{a(t)}{a(t-1)} \\ &= d_t \cdot \frac{1}{1 - d_t} \\ &= \frac{d_t}{1 - d_t} \end{aligned}$$

Relationship between i_t and d_t cont.

A quick picture will help you remember this relationship (and also d_t in terms of i_t). Recall i_t is a measure of interest paid at the end of the period and d_t is a measure of interest paid at the beginning of the period.

$$\begin{array}{ccc} & \text{(A)} & \\ d_t & \longrightarrow & i_t \\ 0 & \longleftarrow & 1 \\ & \text{(B)} & \end{array}$$

$$\text{(A):} \quad \frac{d_t}{1-d_t} = i_t$$

$$\text{(B):} \quad \frac{i_t}{1+i_t} = d_t$$

Relationship between i_t and d_t cont.

Example

Given $i_1 = 12\%$ and $d_2 = 12\%$, find d_1 and i_2.

$$\begin{array}{ccc} d_t & \longrightarrow & i_t \\ 0 & \longleftarrow & 1 \end{array}$$

$$d_1 = \frac{0.12}{1.12} = 10.7\%$$

$$i_2 = \frac{0.12}{1-0.12} = 13.6\%$$

Terminology Note

Note the difference in these two sentences:

1. Cash flows are discounted at an annual effective rate of 10%.
2. The annual effective discount rate is 10%.

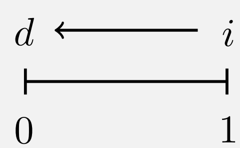
You should interpret these as:

1. $i = 10\%$
2. $d = 10\%$

In the first sentence discount is a *verb* and in the second sentence discount is an *adjective* describing the type of rate.

Exercise 1

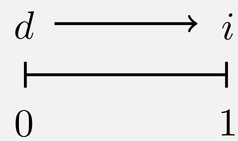
Given $i = 12.5\%$, find d.



$$d = \frac{0.125}{1.125} = \boxed{11.11\%}$$

Exercise 2

Given $d = 5\%$, find i.



$$i = \frac{0.05}{1 - 0.05} = \boxed{5.26\%}$$

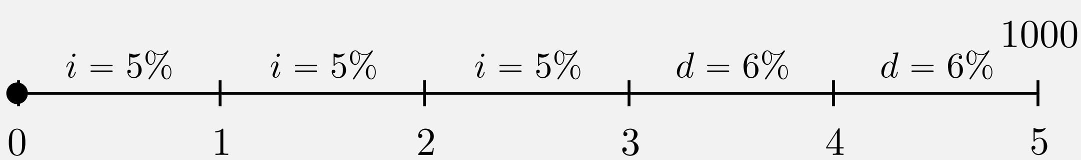
Exercise 3

You are given:

i. For years one through three, cash flows are discounted at an annual effective rate of 5%.

ii. For years four and five the annual effective discount rate is 6%.

Find the present value of 1000 paid five years from now.



$$\frac{1000(1 - 0.06)(1 - 0.06)}{1.05(1.05)(1.05)} = \boxed{763.29}$$

A.1.5 Compound Interest

What is compound interest?

Derive $a(t)$

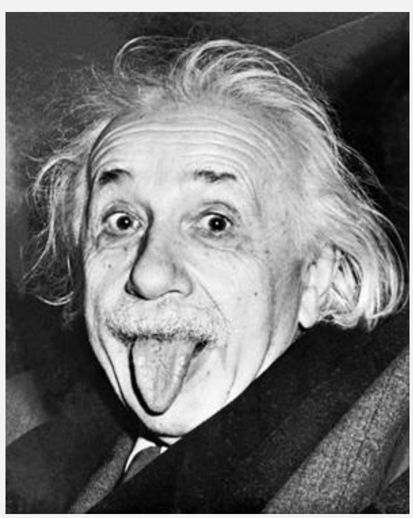
Terminology Notes

Introduction to Calculators

Exercises

Compound Interest

"Compound interest is the eighth wonder of the world. He who understands it, earns it ... he who doesn't ... pays it."



What is compound interest?

compound interest – interest earned also earns interest

100 is borrowed at a compound interest rate of 10% per annum. Find the loan balance after 3 years.

$$
\begin{aligned}
\text{Year 1:}\quad & \text{interest earned} = 10\%(100) = 10 \\
& \text{balance}(1) = 100 + 10 = 110 \\
\text{Year 2:}\quad & \text{interest earned} = 10\%(110) = 11 \\
& \text{balance}(2) = 110 + 11 = 121 \\
\text{Year 3:}\quad & \text{interest earned} = 10\%(121) = 12.1 \\
& \text{balance}(3) = 121 + 12.1 = \boxed{133.1}
\end{aligned}
$$

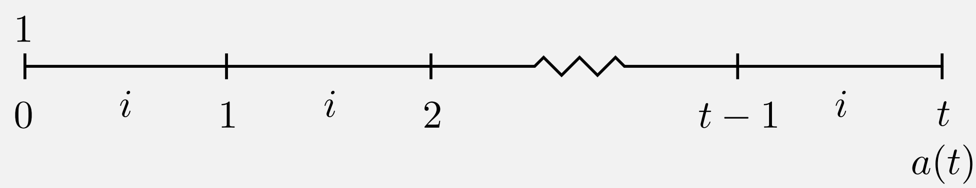
alternatively

$$100(1.10)(1.10)(1.10) = 100(1.10)^3 = 133.1$$

Compound interest is defined using the effective interest rate. Unless stated otherwise, the rate of interest will be the same each period.

Derive $a(t)$

Given compound interest of i per annum, derive $a(t)$.



$$a(1) = 1(1+i) = (1+i)^1$$

$$a(2) = (1+i)(1+i) = (1+i)^2$$

$$a(3) = (1+i)^2(1+i) = (1+i)^3$$

$$a(t) = (1+i)^t$$

Notice we drop the subscript t because $i_t = i$ for all t. Likewise $d_t = d$ and $v_t = v$ for all t .

Terminology Notes

Unless stated otherwise:

- assume compound interest
- assume all interest rates are per annum

For example:

- If the problem says "i = 0.05"

  ⇒ compound interest of 5% per annum

  ⇒ $a(t) = 1.05^t$

  ⇒ effective rate of interest is 5% for every year

Example

Given $d = \frac{1}{9}$, find $a(t)$.

$$i = \frac{d}{1-d} = \frac{\frac{1}{9}}{1-\frac{1}{9}} = 0.125$$

$$a(t) = (1+i)^t = 1.125^t$$

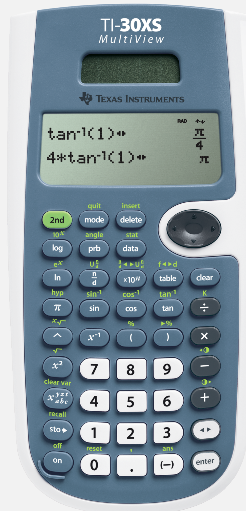
Financial Calculator

For FM you **must** have one of these calculators:

On amazon.com, BA II Plus is \$28.50 and the BA II Plus Professional is \$43.25. There is no computational advantage to the pro version.

MultiView Calculator

You can also use this calculator (in tandem with the financial calculator):



This is the ideal calculator for all other actuarial exams. $16 on amazon.com.

I like to use both the MultiView and one of the financial calculators. I highly recommend you get both.

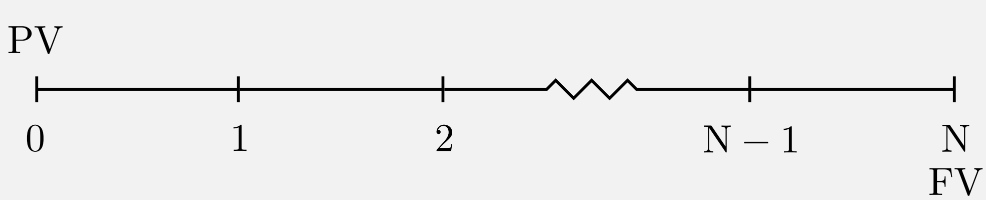
Introduction to Calculator

Switch to calculator and demonstrate the following:

- how to reset the calculator
- chain mode vs algebraic operating system (AOS) mode
- check default P/Y and C/Y
- time value of money keys

TVM keys

N - number of periods
I/Y - effective rate of interest per period (as a percentage)
PV - present value
PMT - payment per period (ignore for now)
FV - future value (or accumulated value)



Introduction to Calculator cont.

Re-work the example from slide 3 using the TVM keys.

100 is borrowed at a compound interest rate of 10% per annum. Find the loan balance after 3 years.

$$\boxed{\begin{aligned} N &= 3 \\ I/Y &= 10 \\ PV &= 100 \\ PMT &= 0 \\ CPT\ FV &= -133.1 \end{aligned}}$$

Why does it return *negative* 133.1?

- The calculator works on a cash inflow/outflow basis.
- Think of positive numbers as money you receive.
- Think of negative numbers as money you pay.
- We received (borrowed) 100 at time 0, thus we must pay back 133.1 at time 3.

Exercise 1

\$100 invested for 3 years, at an effective rate of interest i, will earn \$36 of interest.

Find the accumulated value of \$50 invested at the same rate of compound interest i for 5 years.

$$136 = 100(1+i)^3$$

$$i = \left(\tfrac{136}{100}\right)^{1/3} - 1$$

$$i = 0.10793$$

The AV of \$50 invested for 5 years at i is

$$50(1.10793)^5 = \boxed{83.47}$$

Alternatively

$$50(1+i)^5 = 50\left[(1+i)^3\right]^{\frac{5}{3}} = 50\left[1.36\right]^{\frac{5}{3}} = 83.47$$

Exercise 2

At a certain rate of compound interest:

(i) 1 grows to 3 in x years

(ii) 3 grows to 14 in y years

(iii) 1 grows to 21 in z years

Determine what 5 grows to in $z - x - y$ years.

The three bullet points tell us:

(i) $1(1+i)^x = 3$

(ii) $3(1+i)^y = 14 \Rightarrow (1+i)^y = \frac{14}{3}$

(iii) $1(1+i)^z = 21$

We want to know

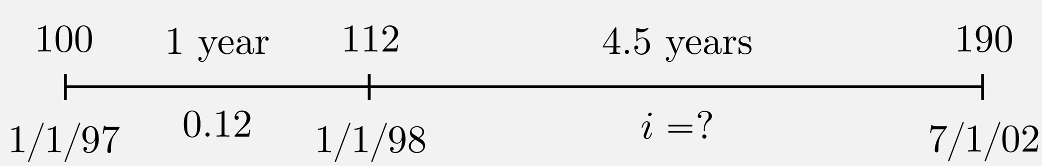
$$5(1+i)^{z-x-y} = \frac{5(1+i)^z}{(1+i)^x \cdot (1+i)^y} = \frac{5(21)}{3 \cdot \frac{14}{3}} = \boxed{7.5}$$

Exercise 3

You invested 100 on January 1, 1997. The investment was worth 190 on July 1, 2002. The effective rate of return for the first year was 12%.

Determine the annualized effective rate of return from January 1, 1998 to July 1, 2002.

Let i be the annual effective rate of return from 1/1/98 to 7/1/02.



After one year the investment is worth $100(1.12) = 112$.

$$190 = 112(1+i)^{4.5}$$

$$i = \boxed{0.1246}$$

A.1.6 Simple Interest

What is Simple Interest?

Derive $a(t)$

Declining Effective Rate of Interest

Additional Notes on Simple Interest

Exercises

What is Simple Interest?

simple interest – interest rate applies to original investment only

100 is invested at simple interest of 10% per annum. Find the accumulated value after 3 years.

$$\begin{aligned}
\text{Year 1:}\quad & \text{interest earned} = 10\%(100) = 10 \\
& \text{balance}(1) = 100 + 10 = 110 \\
\text{Year 2:}\quad & \text{interest earned} = 10\%(100) = 10 \\
& \text{balance}(2) = 110 + 10 = 120 \\
\text{Year 3:}\quad & \text{interest earned} = 10\%(100) = 10 \\
& \text{balance}(3) = 120 + 10 = \boxed{130}
\end{aligned}$$

alternatively

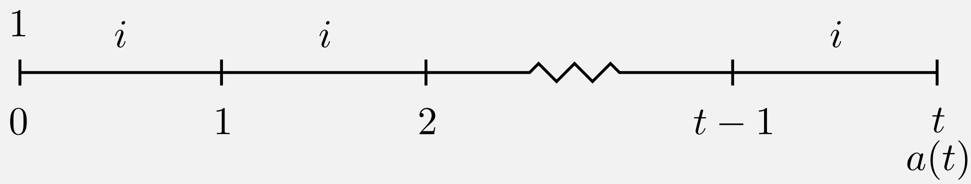
$$100 + 10\%(100)(3) = 130$$

simple interest – **amount of interest** each period is constant

compound interest – **effective interest rate** each period is constant

Derive $a(t)$

Given simple interest of i, find $a(t)$.



$$a(1) = 1 + i(1) = 1 + i$$

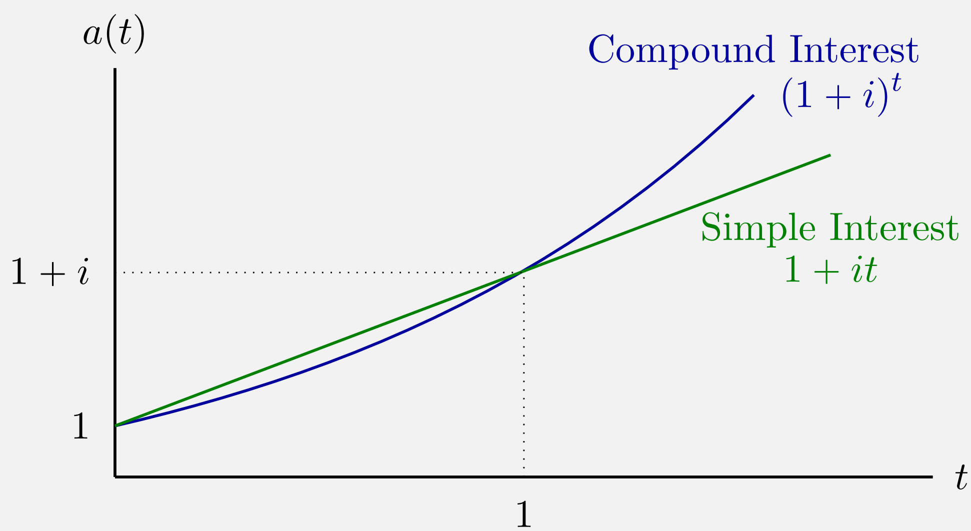
$$a(2) = 1 + i + i(1) = 1 + 2i$$

$$a(3) = 1 + 2i + i(1) = 1 + 3i$$

$$a(t) = 1 + it$$

It is common to use i for simple interest like we used i for compound interest. But note that for simple interest of i per annum, the effective rate of interest is **not** i for all years.

Derive $a(t)$ cont.



For less than 1 year, borrower prefers compound interest of i vs. simple interest of i.

For more than 1 year, lender prefers compound interest of i vs. simple interest of i.

Declining Effective Rate of Interest

Recall from the example on slide 2:
100 is invested at simple interest of 10% per annum.

$$AV(1) = 110 \qquad AV(2) = 120 \qquad AV(3) = 130$$

Thus the annual effective rates of interest are:

$$i_1 = \tfrac{110-100}{100} = 10\%$$

$$i_2 = \tfrac{120-110}{110} = 9.09\%$$

$$i_3 = \tfrac{130-120}{120} = 8.33\%$$

For simple interest of i per annum:

$$i_n = \frac{i}{1 + i(n-1)}$$

thus the annual effective rate of interest decreases each year under simple interest.

Additional Notes on Simple Interest

Simple interest is sometimes used for partial years.

Example

Given $i = 0.10$ and simple interest is used only for partial years. Find the accumulated value of 100 five and half years from now.

The balance 5 years from is

$$100(1.10)^5 = 161.05$$

and then using simple interest for the last half a year, the balance after 5.5 year is

$$161.05(1 + 0.1(0.5)) = \boxed{169.10}$$

If we used compound interest for the entire 5.5 years the balance would be

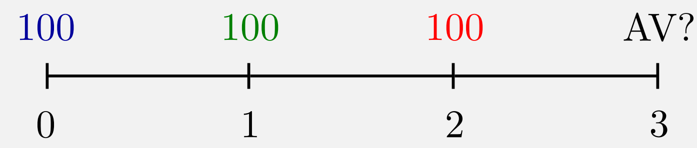
$$100(1.10)^{5.5} = 168.91$$

Additional Notes on Simple Interest cont.

Under simple interest every cash flow is treated separately with its own “time 0”.

Example

100 is deposited at the beginning of each year for 3 years. If the deposits earn simple interest of 10%, what is the accumulated value at the end of 3 years?



Payment	Accumulated Value
1	$100(1+0.1(3)) = 130$
2	$100(1+0.1(2)) = 120$
3	$100(1+0.1(1)) = 110$
Total	$130+120+110 = 360$

Additional Notes on Simple Interest cont.

The wrong way to approach the problem is ...

Example

100 is deposited at the beginning of each year for 3 years. If the deposits earn simple interest of 10%, what is the accumulated value at the end of 3 years?

$$a(t) = 1 + 0.1t$$
$$a(1) = 1 + 0.1(1) = 1.1$$
$$a(2) = 1 + 0.1(2) = 1.2$$
$$a(3) = 1 + 0.1(3) = 1.3$$

$$\text{AV} = 100 \cdot a(3) + 100 \cdot \frac{a(3)}{a(1)} + 100 \cdot \frac{a(3)}{a(2)}$$
$$= 356.52$$

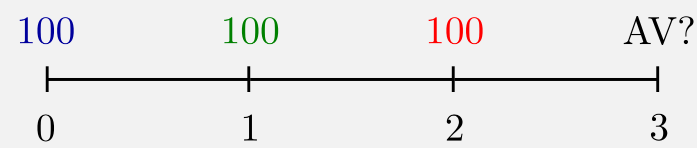
Additional Notes on Simple Interest cont.

The previous solution would be correct if the problem was worded this way ...

Example

---

100 is deposited at the beginning of each year for 3 years. Given $a(t) = 1 + 0.1t$, what is the accumulated value at the end of 3 years?



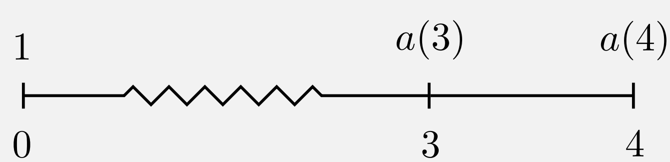
$$a(t) = 1 + 0.1t$$
$$a(1) = 1 + 0.1(1) = 1.1$$
$$a(2) = 1 + 0.1(2) = 1.2$$
$$a(3) = 1 + 0.1(3) = 1.3$$

$$\text{AV} = 100 \cdot a(3) + 100 \cdot \frac{a(3)}{a(1)} + 100 \cdot \frac{a(3)}{a(2)}$$
$$= 356.52$$

Exercise 1

A loan is made for five years at a simple interest rate of 12% per annum. What is the equivalent annual effective rate of discount during the fourth year of the loan?

Let the original loan amount be 1.



The loan balances are

$$a(3) = 1 + 0.12(3) = 1.36$$
$$a(4) = 1 + 0.12(4) = 1.48$$

The effective rate of discount is

$$d_4 = \frac{1.48 - 1.36}{1.48} = \boxed{8.108\%}$$

Exercise 2

At a rate of simple interest i, 10 will accumulate to 15 after x years. What will 20 accumulate, at a simple rate of $2i$, to after $5x$ years?

We are given

$$10(1+ix) = 15$$
$$ix = 0.5$$

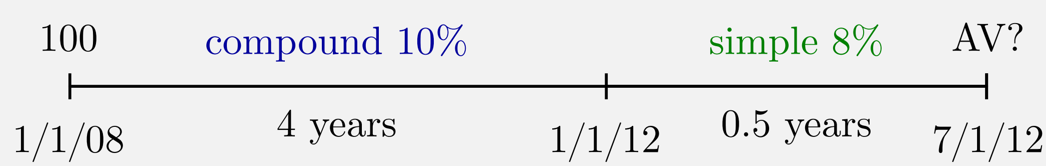
now find the accumulated value of the 20 at $2i$ after $5x$ years

$$\begin{aligned} 20(1+2i(5x)) &= 20(1+10ix) \\ &= 20(1+10(0.5)) \\ &= \boxed{120} \end{aligned}$$

Exercise 3

An investment earns 10% compound interest for each complete year and 8% simple interest for each partial year.

A \$100 investment is made on January 1, 2008. What is the accumulated value of the investment on July 1, 2012?



$$\text{AV} = 100(1.10)^4(1+0.08(0.5)) = \boxed{152.27}$$

A.1.7 Nominal Annual Rates of Interest

What are Nominal Annual Rates of Interest?

Interest rates are usually quoted on the same basis as the payment period.

- e.g. you are quoted a mortgage rate of 6%
- Does this mean $i = 6\%$? No.
- Mortgage payments are payable monthly.
- Nominal annual rate (NAR) convertible monthly is 6%.

$i^{(m)}$ - NAR convertible (or compounded) m times per year

Nominal rates are interest rates in name only (hence “nominal”). In other words, you can not use the nominal rate without first converting it to an effective rate.

$\frac{i^{(m)}}{m}$ – effective rate of interest per m-th of a year

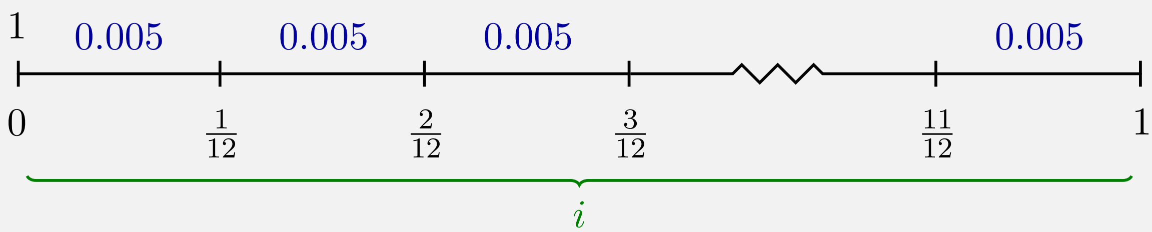
In our mortgage example:

- $i^{(12)} = 6\%$
- $\frac{i^{(12)}}{12} = \frac{0.06}{12} = 0.005 =$ effective rate of interest per month

Equivalent Annual Effective Rate

If $i^{(12)} = 6\%$, what is the equivalent annual effective rate of interest i?

$$\frac{i^{(12)}}{12} = \frac{0.06}{12} = 0.005 = \text{ effective rate per month}$$



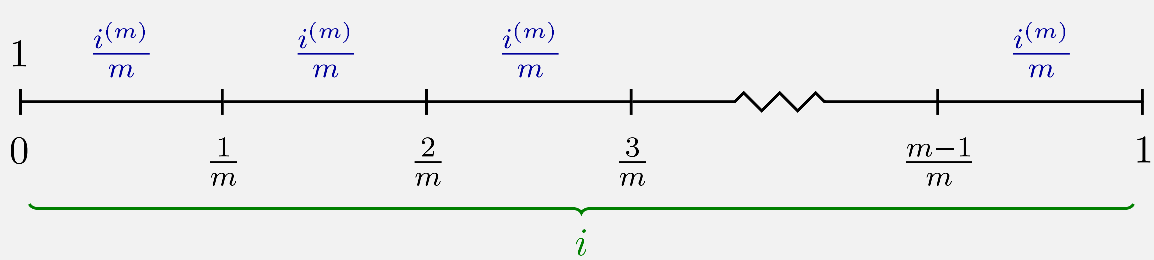
$$1(1.005)^{12} = 1(1+i)$$
$$i = 0.0616778$$

In the financial world, NAR is called APR (annual percentage rate). A loan at an APR of 6% sounds like a better deal than a loan at an effective rate of 6.16778%, but they are equivalent.

Equivalent Annual Effective Rate cont.

Given $i^{(m)}$, what is the equivalent annual effective rate of interest i?

$$\frac{i^{(m)}}{m} = \text{ effective rate per } m\text{-th of a year}$$



$$1\left(1+\frac{i^{(m)}}{m}\right)^m = 1(1+i) \quad \Rightarrow \quad i = \left(1+\frac{i^{(m)}}{m}\right)^m - 1$$

$$\Rightarrow \quad i^{(m)} = m\left[(1+i)^{\frac{1}{m}} - 1\right]$$

Equivalent Annual Effective Rate cont.

Given $i = 12\%$, find the equivalent rates $i^{(2)}$, $i^{(4)}$ and $i^{(12)}$.

You could simply plug into the formula from the previous slide:

i. $i^{(2)} = 2\left[(1.12)^{\frac{1}{2}} - 1\right] = 0.1166$

ii. $i^{(4)} = 4\left[(1.12)^{\frac{1}{4}} - 1\right] = 0.1149$

iii. $i^{(12)} = 12\left[(1.12)^{\frac{1}{12}} - 1\right] = 0.1139$

but I prefer to **think** through these on the calculator.

i. 0.12 is the annual effective rate of interest
ii. add 1 = 1.12 is the accumulation factor for one year
iii. raise to the 1/2 power = 1.0583 is the accumulation factor for 6 months
iv. subtract 1 = 0.0583 is the effective rate per 6 months
v. multiply 2 = 0.1166 is the NAR convertible twice per year

$$i^{(\infty)} < \cdots < i^{(12)} < i^{(4)} < i^{(2)} < i$$

Converting between $i^{(m)}$ and $i^{(n)}$

Derive relationship between $i^{(m)}$ and $i^{(n)}$.

Accumulate 1 for one year using both rates:

$$1\left(1 + \frac{i^{(n)}}{n}\right)^n = 1\left(1 + \frac{i^{(m)}}{m}\right)^m$$

$$1 + \frac{i^{(n)}}{n} = \left(1 + \frac{i^{(m)}}{m}\right)^{\frac{m}{n}}$$

$$i^{(n)} = n\left[\left(1 + \frac{i^{(m)}}{m}\right)^{\frac{m}{n}} - 1\right]$$

Example

Given $i^{(12)} = 0.12$, find the equivalent rate $i^{(4)}$.

$$i^{(4)} = 4\left[(1 + \tfrac{0.12}{12})^{\frac{12}{4}} - 1\right] = \boxed{0.121204}$$

A solution thinking through the conversion on next slide.

Converting between $i^{(m)}$ and $i^{(n)}$ cont.

Given $i^{(12)} = 0.12$, find the equivalent rate $i^{(4)}$.

i. 0.12 is the nominal annual rate convertible monthly
ii. divide by 12 = 0.01 is the effective rate per month
iii. add 1 = 1.01 is the accumulation factor per month
iv. raise to 3 = 1.030301 is the accumulation factor per quarter
v. subtract 1 = 0.030301 is the effective rate per quarter
vi. multiply by 4 = $\boxed{0.121204}$ is the NAR convertible quarterly

Rates Convertible Less Frequently Than Annually

The nominal annual rate convertible once every two years is 15%. Find the accumulated value of 1000 in 5 years.

$$i^{(\frac{1}{2})} = 0.15$$

$$\frac{i^{\left(\frac{1}{2}\right)}}{\frac{1}{2}} = \frac{0.15}{\frac{1}{2}} = 0.30 = \text{effective rate per 2 years}$$

That is too tedious to write. Instead I usually do something like ...

$$\text{Let } j = \text{effective rate per 2 years}$$
$$j = \tfrac{0.15}{1/2} = 0.30$$

In fact, I will usually do this with normal frequencies too (monthly, quarterly, semi-annually).

$$\text{AV} = 1000(1 + 0.30)^{5/2} = \boxed{1926.90}$$

Calculator Notes

I said in a previous lesson that I/Y is the effective rate per period. It is really the NAR, but if you keep C/Y = 1, then I/Y is the effective rate per period. Always keep C/Y = P/Y = 1! It will make your life easier. I feel so strongly about this that I will not show how to work problems changing C/Y.

Example

You are given $i^{(12)} = 10\%$. Find the present value of 100 paid 9 years from now. 9 years = 9 years × 12 months per year = 108 months

$$\boxed{\begin{aligned} N &= 108 \\ I/Y &= 10/12 \\ PMT &= 0 \\ FV &= 100 \\ \text{CPT } PV &= -40.81 \end{aligned}}$$

The present value is $\boxed{40.81}$.

Alternatively: $100\left(1+\frac{0.10}{12}\right)^{-9(12)} = 40.81.$

Exercise 1

Convert the following rates using only your calculator (no pen or paper):

Given	Find	Answer
$i = 0.08$	$i^{(12)}$	0.07721
$i^{(2)} = 0.06$	i	0.06090
$i^{(4)} = 0.10$	$i^{(2)}$	0.10125

Exercise 2

Given $i^{(2)} = 0.08$, find the accumulated value of 500 in 4.5 years.

The effective rate per six months $= 0.08/2 = 0.04$.

500 accumulated for 4.5 years is

$$500\,(1 + 0.04)^{4.5(2)} = \boxed{711.66}$$

Alternatively you can use the TVM function.

$$\boxed{\begin{aligned} N &= 9 \\ I/Y &= 4 \\ PV &= 500 \\ PMT &= 0 \\ CPT\ FV &= -711.6559062 \end{aligned}}$$

Exercise 3

The nominal annual interest rate convertible once every 4 years is 6%. Find the present value of 400 to be paid in 12 years.

The effective rate per 4 years $= 0.06/(1/4) = 0.06(4) = 0.24$.

The present value of 400 paid in 12 years is

$$400\,(1.24)^{-12/4} = \boxed{209.79}$$

Alternatively you can use the TVM function.

$$\boxed{\begin{aligned} N &= 3 \\ I/Y &= 24 \\ PMT &= 0 \\ FV &= 400 \\ CPT\ PV &= -209.7949045 \end{aligned}}$$

A.1.8 Nominal Annual Rates of Discount

Nominal Rates of Discount

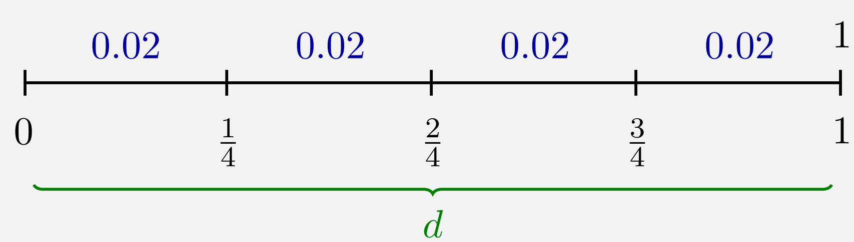
$d^{(m)}$ – NAR of discount convertible m times per year

$\frac{d^{(m)}}{m}$ – effective rate of discount per m-th of a year

Example

Given $d^{(4)} = 0.08$, find d.

$$\frac{d^{(4)}}{4} = \frac{0.08}{4} = 0.02 = \text{ effective discount rate per quarter}$$



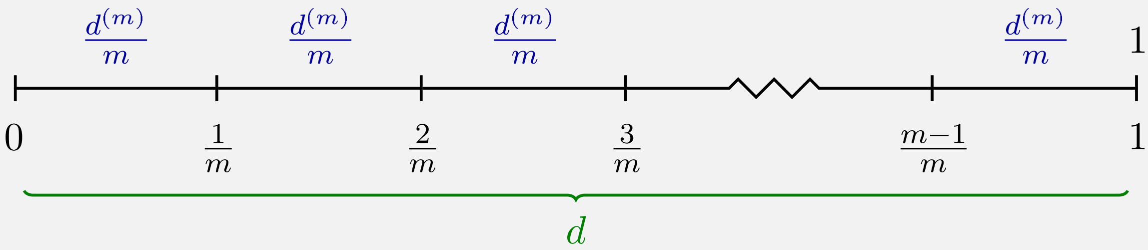
$$1(1-0.02)^4 = 1(1-d)$$

$$d = \boxed{0.07763}$$

Nominal Rates of Discount cont.

Write d in terms of $d^{(m)}$ and vice versa.

$$\frac{d^{(m)}}{m} = \text{effective rate of discount per } m\text{-th of a year}$$



$$1\left(1-\frac{d^{(m)}}{m}\right)^m = 1(1-d) \quad \Rightarrow \quad d = 1-\left(1-\frac{d^{(m)}}{m}\right)^m$$

$$\Rightarrow \quad d^{(m)} = m\left[1-(1-d)^{\frac{1}{m}}\right]$$

Nominal Rates of Discount cont.

Given $i = 0.12$, find $d^{(2)}$, $d^{(4)}$ and $d^{(12)}$.

$$d = \frac{0.12}{1.12} = 0.10714$$

You could simply plug into the formula from the previous slide:

i. $d^{(2)} = 2\left[1-(1-0.10714)^{\frac{1}{2}}\right] = 0.1102$

ii. $d^{(4)} = 4\left[1-(1-0.10714)^{\frac{1}{4}}\right] = 0.1117$

iii. $d^{(12)} = 12\left[1-(1-0.10714)^{\frac{1}{12}}\right] = 0.1128$

but I prefer to **think** through these on the calculator.

i. 0.12 is the annual effective rate of interest
ii. add 1 = 1.12 is the accumulation factor for one year
iii. raise to -1 = 0.892857 is the discount factor for one year
iv. raise to the 1/12 power = 0.9906 is the discount factor for 1 month
v. subtract from 1 = 0.0094 is the effective discount rate per month
vi. multiply 12 = 0.1128 is the nominal annual discount rate convertible monthly

$$d < d^{(2)} < d^{(4)} < d^{(12)} < \cdots < d^{(\infty)} = i^{(\infty)} < \cdots < i^{(12)} < i^{(4)} < i^{(2)} < i$$

Convert between $d^{(m)}$ and $d^{(n)}$

Derive relationship between $d^{(m)}$ and $d^{(n)}$.

Discount 1 for one year using both rates:

$$1\left(1-\frac{d^{(n)}}{n}\right)^{n} = 1\left(1-\frac{d^{(m)}}{m}\right)^{m}$$

$$1-\frac{d^{(n)}}{n} = \left(1-\frac{d^{(m)}}{m}\right)^{\frac{m}{n}}$$

$$d^{(n)} = n\left[1-\left(1-\frac{d^{(m)}}{m}\right)^{\frac{m}{n}}\right]$$

Example

Given $d^{(2)} = 0.12$, find the equivalent rate $d^{(4)}$.

$$d^{(4)} = 4\left[1-(1-\tfrac{0.12}{2})^{\frac{2}{4}}\right] = \boxed{0.121856}$$

A solution thinking through the conversion on next slide.

Nominal Rates of Discount cont.

Given $d^{(2)} = 0.12$, find the equivalent rate $d^{(4)}$.

i. 0.12 is the nominal discount rate convertible semi-annually
ii. divide by 2 = 0.06 is the effective discount rate per 6 months
iii. subtract from 1 = 0.94 is the discount factor for 6 months
iv. raise to 1/2 = 0.969535971 is the discount factor for 3 months (one quarter)
v. subtract from 1 = 0.030464029 is the effective discount rate per quarter
vi. multiply by 4 = $\boxed{0.121856}$ is the nominal discount rate convertible quarterly

Converting between $i^{(n)}$ and $d^{(m)}$

Given $i^{(12)} = 0.12$, find the equivalent rate $d^{(2)}$.

i. 0.12 is the nominal rate convertible monthly
ii. divide by 12 = 0.01 is the effective rate per month
iii. add 1 = 1.01 is the accumulation factor per month
iv. raise to 6 = 1.061520151 is the accumulation factor for 6 months
v. raise to -1 = 0.942045235 is the discount factor for 6 months
vi. subtract from 1 = 0.057954765 is the effective discount rate for 6 months
vii. multiply by 2 = $\boxed{0.1159}$ is the nominal discount rate convertible semi-annually

If you want a formula, here it is

$$\left(1 + \frac{i^{(n)}}{n}\right)^n = \left(1 - \frac{d^{(m)}}{m}\right)^{-m}$$

Both sides accumulate 1 for one year.

Calculator Notes

It is possible to compute nominal annual rates using ICONV. Personally I would never use ICONV, but here is how you do it.

Rework previous examples using ICONV

Given $i^{(12)} = 0.12$, find the equivalent rate $i^{(4)}$.

2nd ICONV
NOM = 12
C/Y = 12
CPT EFF = 12.6825
C/Y = 4
CPT NOM = 12.1204

Given $d^{(2)} = 0.12$, find the equivalent rate $d^{(4)}$.

2nd ICONV
NOM = −12
C/Y = 2
CPT EFF = −11.64
C/Y = 4
CPT NOM = −12.1856

To convert from $i^{(n)}$ to $d^{(m)}$ using ICONV you have to (1) convert $i^{(n)}$ to i, (2) convert i to d, and (3) convert d to $d^{(m)}$. Example, $i^{(12)} = 0.12$, find $d^{(2)}$.

Exercise 1

Convert the following rates using only your calculator (no pen or paper):

Given	Find	Answer
$d = 0.04$	$d^{(4)}$	0.04061
$d = 0.12$	i	0.13636
$d^{(2)} = 0.07$	$i^{(12)}$	0.07147
$i^{(4)} = 0.10$	$d^{(2)}$	0.09637

Exercise 2

Given $d^{(2)} = 0.08$, find the accumulated value of 500 in 4.5 years.

$$500\left(1 - \frac{0.08}{2}\right)^{-4.5(2)} = \boxed{721.99}$$

Alternatively you can use the TVM function.

$$\boxed{\begin{aligned} \text{N} &= 9 \\ \text{I/Y} &= 4.16667 \\ \text{PV} &= 500 \\ \text{PMT} &= 0 \\ \text{CPT FV} &= -721.99 \end{aligned}}$$

i. 0.08 is the nominal rate of discount convertible semi-annually
ii. divide by 2 = 0.04 is effective rate of discount per 6 months
iii. subtract from 1 = 0.96 is the discount factor per 6 months
iv. raise to -1 = 1.0416667 is the accumulation factor per 6 months
v. subtract 1 = 0.041667 is the effective rate of interest per 6 months

Exercise 3

The nominal annual discount rate convertible once every 4 years is 6%. Find the present value of 400 to be paid in 10 years.

$$400\left(1-\frac{0.06}{1/4}\right)^{10/4} = \boxed{201.42}$$

Alternatively you can use the TVM function.

$$\boxed{\begin{aligned} \text{N} &= 10 \\ \text{I/Y} &= 7.10176 \\ \text{PMT} &= 0 \\ \text{FV} &= 400 \\ \text{CPT PV} &= -201.42 \end{aligned}}$$

i. 0.06 is the nominal rate of discount convertible once every 4 years
ii. divide by 1/4 = 0.24 is the effective rate of discount per 4 years
iii. subtract from 1 = 0.76 is the discount factor per 4 years
iv. raise to -1 = 1.315789474 is the accumulation factor per 4 years
v. raise to 1/4 = 1.0710176 is the accumulation factor for 1 year
vi. subtract 1 = 0.0710176 is the effective rate of interest per year

A.1.9 Force of Interest – Part 1

What is the Force of Interest?

Accumulation Function

Discount Function

What is the Force of Interest?

All the measurements of interest in the preceding lessons are useful for measuring interest over specified intervals of time (e.g. a year, 6 months, quarterly, or monthly).

Also necessary to measure the intensity of interest for each moment in time. In other words, over infinitesimally small intervals of time.

This measure of interest is called the **force of interest**.

The force of interest at time t is denoted δ_t.

1. δ_t is a measure of the intensity of interest at exact time t.
2. δ_t expresses this measurement as a rate per measure period (one year unless stated otherwise).

What is the Force of Interest? cont.

Consider a fund with a balance of $A(t)$ at time t.

- The only force acting on the fund is interest.
- The rate of change on $A(t)$ is measured by the slope.
- The slope of a curve is found by taking the first derivative.
- $A'(t)$ is not sufficient for the measure of intensity because the measure of intensity must not depend on the amount invested.
  - e.g. if 1000 and 500 are invested in the same fund, the rate of change will be twice as great for the 1000 investment, but interest is not operating with twice the intensity.
- We divide by $A(t)$ to to adjust for the amount invested at time t (just like we divide by $A(t)$ for the effective rate of interest).

$$\delta_t = \frac{A'(t)}{A(t)} = \frac{a'(t)}{a(t)}$$

Accumulation Function

Given $\delta_t = \dfrac{a'(t)}{a(t)}$, find $a(n)$ in terms of δ_t.

$$\delta_t = \frac{a'(t)}{a(t)}$$

$$\delta_t = \tfrac{d}{dt} \ln a(t)$$

$$\int_0^n \delta_t \, dt = \int_0^n \tfrac{d}{dt} \ln a(t) \, dt$$

$$\int_0^n \delta_t \, dt = \ln a(n) - \ln a(0) = \ln a(n)$$

$$a(n) = \exp\left[\int_0^n \delta_t \, dt\right]$$

In words: to accumulate from time 0 to n, add up the force of interest from time 0 to n and take the exponential.Note n does not have to be an integer.

Accumulation Function cont.

Let $a(t_1, t_2)$ denote the accumulation factor from time t_1 to t_2. In other words, to accumulate from time t_1 to t_2 you multiply by $a(t_1, t_2)$.

$$a(t_1, t_2) = \frac{a(t_2)}{a(t_1)} = \frac{\exp\left[\int_0^{t_2} \delta_t \, dt\right]}{\exp\left[\int_0^{t_1} \delta_t \, dt\right]}$$

$$= \exp\left[\int_0^{t_2} \delta_t \, dt - \int_0^{t_1} \delta_t \, dt\right]$$

$$= \exp\left[\int_{t_1}^{t_2} \delta_t \, dt\right]$$

In words: to accumulate from time t_1 to t_2, add up the force of interest from time t_1 to t_2 and take the exponential (same as previous slide).

Accumulation Function Example

If $\delta_t = 0.15\sqrt{t}$ and an amount of 5000 is invested at time $t = 1$, what is the accumulated value at time $t = 4$?

$$5000\, a(1,4) = 5000 \exp\left[\int_1^4 \delta_t \, dt\right] = 5000 \exp\left[\int_1^4 0.15 t^{0.5} \, dt\right]$$

$$= 5000 \exp\left[\tfrac{0.15}{1.5} t^{1.5} \Big|_1^4\right]$$

$$= \boxed{10{,}068.76}$$

Discount Function

$a(t)$ is the accumulation function. It accumulates cash flows from time 0 to t.

$\dfrac{1}{a(t)}$ is the discount function. It discounts cash flows from time t to 0.

The discount function is also written as $a^{-1}(t)$. This is common notation for the inverse of a function, but in financial mathematics it is used to denote the discount function.

$$a^{-1}(t) \equiv \frac{1}{a(t)} = \frac{1}{\exp\left[\int_0^t \delta_t\,dt\right]} = \exp\left[-\int_0^t \delta_t\,dt\right]$$

To accumulate from time 0 to t, we add up the force of interest from time 0 to t and take the exponential. To discount from time t to time 0, we add up the force of interest from time 0 to t, **take the negative** and take the exponential.

Discount Function cont.

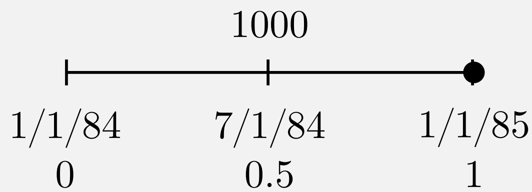
Let $a^{-1}(t_1, t_2)$ denote the discount factor from time t_2 to t_1. In other words, to discount from time t_2 to t_1 you multiply by $a^{-1}(t_1, t_2)$.

$$\begin{aligned} a^{-1}(t_1,t_2) &= \frac{1}{a(t_1,t_2)} \\ &= \frac{1}{\exp\left[\int_{t_1}^{t_2} \delta_t\,dt\right]} \\ &= \exp\left[-\int_{t_1}^{t_2} \delta_t\,dt\right] \end{aligned}$$

In words: to discount from time t_2 to t_1, add up the force of interest from time t_1 to t_2, take the negative and take the exponential.

Exercise 1

On July 1, 1984, a person invested \$1000 in a fund for which the force of interest at time t is given by $\delta_t = (3+2t)/50$, where t is the number of years since January 1, 1984. Determine the accumulated value of the investment on January 1, 1985.



$$
\begin{aligned}
1000\,a(0.5,1) &= 1000\exp\left[\int_{0.5}^{1}\frac{3+2t}{50}\,dt\right]\\
&= 1000\exp\left[\tfrac{1}{50}\left(3t+t^2\right)\Big|_{0.5}^{1}\right]\\
&= \boxed{1046.03}
\end{aligned}
$$

Exercise 2

X is deposited into a savings account at time $t = 0$. No other amounts are deposited. The force of interest for the fund is $\delta_t = t/30$. The balance of fund after 10 years is 12,500.

Determine X.

$$
\begin{aligned}
X = 12{,}500\,a^{-1}(10) &= 12{,}500\exp\left[-\int_{0}^{10}\frac{t}{30}\,dt\right]\\
&= 12{,}500\exp\left[-\frac{t^2}{60}\Big|_{0}^{10}\right]\\
&= \boxed{2{,}360.95}
\end{aligned}
$$

Exercise 3

\$1000 is deposited into a savings account at time $t = 0$. No other amounts are deposited. The accumulated amount in the account at time t is given by:

$$A(t) = 1000\left(1 + \frac{2t}{35}\right)^2$$

Determine the force of interest at time $t = 32.5$.

$$\delta_t = \frac{A'(t)}{A(t)} = \frac{1000(2)\left(1 + \frac{2t}{35}\right)\left(\frac{2}{35}\right)}{1000\left(1 + \frac{2t}{35}\right)^2}$$

$$\delta_{32.5} = \frac{2\left(\frac{2}{35}\right)}{\left(1 + \frac{2(32.5)}{35}\right)} = \boxed{0.04}$$

Exercise 4

A deposit of 1 will accumulate to 2.7183 in 10 years with a force of interest

$$\delta_t = \begin{cases} kt & 0 < t \leqslant 5 \\ 0.04kt^2 & 5 < t \leqslant 10 \end{cases}$$

Determine k.

$$a(10) = a(0,5) \cdot a(5,10)$$

$$2.7183 = \exp\left[\int_0^5 kt\,dt\right] \cdot \exp\left[\int_5^{10} 0.04kt^2\,dt\right]$$

$$2.7183 = \exp\left[k \cdot \frac{t^2}{2}\Big|_0^5\right] \cdot \exp\left[0.04k \cdot \frac{t^3}{3}\Big|_5^{10}\right]$$

$$2.7183 = e^{12.5k} \cdot e^{11.667k} = e^{24.167k}$$

$$k = \boxed{0.0414}$$

A.1.10 Force of Interest – Part 2

Shortcut: Force of Interest → Accumulation Function

Sometimes it is easy to look at a given force of interest and immediately write down the accumulation function.

Example

Given $\delta_t = \frac{2t}{t^2+1}$, find $a(t)$.

$a(t) = t^2 + 1$, because

$$\delta_t = \frac{a'(t)}{a(t)} = \frac{2t}{t^2+1}$$

We could have found $a(t)$ the usual way – add up the force of interest from time 0 to t and taking the exponential, but the shortcut is much faster.

$$\begin{aligned} a(t) = \exp\left[\int_0^t \delta_s\, ds\right] = \exp\left[\int_0^t \frac{2s}{s^2+1}\, ds\right] &= \exp\left[\ln(s^2+1)\Big|_0^t\right] \\ &= \exp\left[\ln(t^2+1) - \ln(0^2+1)\right] \\ &= \exp\left[\ln(t^2+1)\right] = t^2+1 \end{aligned}$$

Shortcut cont.

Avoid this trap.

Given $\delta_t = \frac{2t}{t^2+8}$, find $a(t)$.

$a(t) \neq t^2 + 8$, because $a(0) = 8$, but $a(0)$ has to equal 1.

$$\delta_t = \frac{2t}{t^2+8} = \frac{\frac{2t}{8}}{\frac{t^2+8}{8}} = \frac{\frac{1}{4}t}{\frac{1}{8}t^2+1}$$

$\therefore\ a(t) = \frac{1}{8}t^2 + 1.$

Shortcut cont.

What happens if we multiply the force of interest by a factor of k?

Given $\delta_t \Rightarrow a(t)$. If $\delta_t^* = k\delta_t$, what is $a^*(t)$?

$$\begin{aligned} a^*(t) = \exp\left(\int_0^t \delta_s^*\, ds\right) &= \exp\left(\int_0^t k\delta_s\, ds\right) \\ &= \left[\exp\left(\int_0^t \delta_s\, ds\right)\right]^k \\ &= [a(t)]^k \end{aligned}$$

Example

Given $\delta_t = \frac{4}{1+t}$, find $a(t)$.

$$\delta_t = 4\left(\frac{1}{1+t}\right)$$

The accumulation function for $\frac{1}{1+t}$ is $1+t$, $\therefore\ a(t) = (1+t)^4$.

Compound Interest

Given $a(t) = (1+i)^t$, find δ_t.

$$\delta_t = \frac{a'(t)}{a(t)} = \frac{(1+i)^t \ln(1+i)}{(1+i)^t} = \ln(1+i)$$

Because δ_t does not vary with t

$$\begin{aligned} \delta &= \ln(1+i) \\ e^\delta &= 1+i \\ i &= e^\delta - 1 \end{aligned}$$

For a constant force of interest δ

$$\begin{aligned} a(t) = (1+i)^t = (e^\delta)^t \quad &\Rightarrow \quad a(t) = e^{\delta t} \\ a(t_1, t_2) = \exp\left[\int_{t_1}^{t_2} \delta\, dt\right] \quad &\Rightarrow \quad a(t_1, t_2) = e^{\delta(t_2 - t_1)} \\ &\Rightarrow \quad a^{-1}(t_1, t_2) = e^{-\delta(t_2 - t_1)} \end{aligned}$$

Compound Interest cont.

Accumulation Example

If $\delta = 0.05$ and an amount of 5000 is invested at time $t = 1$, what is the accumulated value at time $t = 4$?

$$5000e^{0.05(3)} = \boxed{5809.17}$$

Discounting Example

If $\delta = 0.08$, find the present value of 1000 to be paid in 4.25 years.

$$1000e^{-0.08(4.25)} = \boxed{711.77}$$

Simple Interest

Given $a(t) = 1 + it$, find δ_t.

$$\delta_t = \frac{a'(t)}{a(t)} = \frac{i}{1+it}$$

not constant
decreases over time

Terminology Note

5000 is invested at time $t = 1$, find the accumulated value at time $t = 4$, given:

(i) $\delta_t = \dfrac{0.08}{1+0.08t} \quad \Rightarrow a(t) = 1 + 0.08t$

$$5000 \cdot a(1,4) = 5000 \cdot \tfrac{a(4)}{a(1)} = 5000 \cdot \tfrac{1+0.08(4)}{1+0.08(1)} = 6111.11$$

(ii) Simple interest at a rate of 8%.

$$5000(1 + 0.08(3)) = 6200$$

If the problem explicitly says "simple interest", then each cash-flow has its own "time 0".

Exercise 1

(a) Given $d = 0.02$, find δ.

i. 0.02 is the annual effective discount rate
ii. subtract that from 1 = 0.98 is the discount factor for one year
iii. raise to -1 = 1.020408 is the accumulation factor for one year
iv. take ln = $\boxed{0.0202}$ is the force of interest

(b) Given $\delta = 0.06$, find $i^{(12)}$.

i. 0.06 is the force of interest
ii. take exp = 1.061836547 is the accumulation factor for one year
iii. raise to 1/12 = 1.005012521 is the accum. factor for one month
iv. subtract 1 = 0.005012521 is the effective rate of interest per month
v. multiply by 12 = $\boxed{0.06015}$ is the NAR convertible monthly

Exercise 2

At a force of interest $\delta = 0.05$, the following payments have the same present value:

(i) X at the end of year 5 plus $2X$ at the end of year 10

(ii) Y at the end of year 14

Calculate Y/X.

$$\begin{aligned} Xe^{-5(0.05)} + 2Xe^{-10(0.05)} &= Ye^{-14(0.05)} \\ 1.991862X &= 0.4965853Y \\ Y/X &= \boxed{4.0111} \end{aligned}$$

Exercise 3

On 1/1/97, Kelly deposits X into a bank account. The account is credited with simple interest at a rate of 10% per year.

On the same date, Tara deposits X into a different bank account. The account is credited interest using a force of interest:

$$\delta_t = \frac{2t}{t^2 + k}$$

From the end of the 4th year until the end of the 8th year, both accounts earn the same dollar amount of interest.

Calculate k.

For Kelly: $a_K(t) = 1 + 0.1t$

For Tara: $\delta_t = \dfrac{2t}{t^2 + k} = \dfrac{\frac{2}{k}t}{\frac{1}{k}t^2 + 1} \quad \Rightarrow \quad a_T(t) = \frac{1}{k}t^2 + 1$

Exercise 3 cont.

$$a_K(t) = 1 + 0.1t \qquad a_T(t) = \frac{1}{k}t^2 + 1$$

We are given the dollar amount of interest earned from the end of the 4th year to end of the 8th year is equal for Kelly and Tara.

$$\begin{aligned} Xa_K(8) - Xa_K(4) &= Xa_T(8) - Xa_T(4) \\ 1 + 0.1(8) - (1 + 0.1(4)) &= \tfrac{1}{k}(8)^2 + 1 - (\tfrac{1}{k}(4)^2 + 1) \\ 0.4 &= \tfrac{48}{k} \\ k &= \boxed{120} \end{aligned}$$