THE INFINITE ACTUARY'S

SAMPLE DETAILED STUDY GUIDE FOR THE

QFI Quant Exam

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MFD Chapter 2: A Primer on the Arbitrage Theorem

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Overview of This Reading

This chapter starts by defining arbitrage, which then leads to the formal statement of the arbitrage theorem. This chapter also discusses martingales and binomial pricing.

Key topics for the exam include:

- Define arbitrage
- Apply the arbitrage theorem to numerical examples
- Describe martingales and submartingales
- Describe binomial pricing

Arbitrage

- **Arbitrage** - taking simultaneous positions in different assets so that one guarantees a riskless profit higher than the risk-free rate
  - No net investment, but future positive profit
  - Negative net commitment, but nonnegative profits
- Utilizations of arbitrage-free prices:
  - Pricing a new product in the market
  - Risk management
  - Marking to market (might use arbitrage-free price for illiquid assets)
  - Comparing with market

$S(t)$ will denote a vector of asset prices. Each entry of this vector is the price of a specific financial security at some point in time. For example, $S_1(t)$ may represent borrowing and $S_2(t)$ may represent the value of a share of stock.

$$S_t = \begin{bmatrix} S_1(t) \\ S_2(t) \\ \dots \\ S_n(t) \end{bmatrix}$$

W represents the possible payouts in the future possible states of the world:

$$W = \begin{bmatrix} \omega_1(t) \\ \omega_2(t) \\ \dots \\ \omega_n(t) \end{bmatrix}$$

D_t will be the matrix containing payoffs:

- d_{ij} will denote the number of units of account paid by one unit of security i in state j.
- i is the index for the security (out of N total assets)
- j is the index for the state (out of K total states of the world)

For now, consider $N = 3$ assets: risk-free T-bill, an underlying asset, and a call option. Assume that Δ time elapses and there are two possible future states of the world.

Then $K = 2$ and $N = 3$ and the payoff matrix D_t is given by:

$$D_t = \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1K} \\ d_{21} & d_{22} & \dots & d_{2K} \\ \vdots & \vdots & \vdots & \vdots \\ d_{N1} & d_{N2} & \dots & d_{NK} \end{bmatrix} = \begin{bmatrix} (1+r\Delta) & (1+r\Delta) \\ S_1(t+\Delta) & S_2(t+\Delta) \\ C_1(t+\Delta) & C_2(t+\Delta) \end{bmatrix}$$

The Arbitrage Theorem

Positive constants ψ_1 and ψ_2 can be found such that asset prices satisfy:

$$\begin{bmatrix} 1 \\ S(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} (1+r\Delta) & (1+r\Delta) \\ S_1(t+\Delta) & S_2(t+\Delta) \\ C_1(t+\Delta) & C_2(t+\Delta) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

if and only if there are no-arbitrage possibilities.

The Arbitrage Theorem - Example

Suppose we are given a 10% annual interest rate, and that the stock price is either \$100 or \$150 in the next time step. Assume we want to value a call with strike 100.

We are given the matrix equation below:

$$\begin{bmatrix} 1 \\ 100 \\ C \end{bmatrix} = \begin{bmatrix} 1.1 & 1.1 \\ 100 & 150 \\ 0 & 50 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

- Two equations with two unknowns and a third equation showing $C = 50\psi_2$.
- Thus if we solve for ψ_2 we can get the call premium
- Solving the system of equations $1.1\psi_1 + 1.1\psi_2 = 1$ and $100\psi_1 + 150\psi_2 = 100$ gives:
$$\psi_1 = .7272 \text{ and } \psi_2 = .1818$$
- Thus, $C = \sum_i \psi_i C_i = 50\psi_2 = 50 \cdot .1818 = 9.09$. This is the no-arbitrage price of the call.
- Note that here there is a unique state price vector.

Implications of the Arbitrage Theorem

- Define the state prices ψ_i, which represent how much investors are willing to pay for one unit of money in state i. If state i is realized, the investor gets one unit of money, otherwise nothing.
- This gives us an important equation: $\sum_i \psi_i = v = \frac{1}{1+r}$.
  - Here, I am using v to denote the PV of one unit of money in every state.
- Can define the risk-neutral probabilities through the state prices. We normalize the state prices so they add to one and set $Q_i = (1+r)\psi_i$.
- Then $\sum_i Q_i = 1$, and by the arbitrage theorem we have the strict inequality $0 < Q_i < 1$.
- Can use these Q_i values for risk-neutral pricing.
- For example, $C(t) = \sum_i \psi_i C_i(t+1) = \frac{1}{1+r} \sum_i (1+r)\psi_i C_i(t+1) = \frac{1}{1+r} \sum_i Q_i C_i(t+1)$
- Thus, $C(t) = \frac{1}{1+r} E^Q(C(t+1))$.
- Can rearrange to see we are earning the risk-free rate under measure Q in risk-neutral pricing, $1+r = \frac{E^Q(C(t+1))}{C(t)}$.
- We can value assets using their discounted risk-neutral expected payoff. We used C above, but can also replace with S as well
- Note that these are not necessarily true probabilities, they are just convenient for pricing

State Price Vector Example

Suppose there are three possible future states of the world. You are given that:

- $\psi_1 = .2$
- $\psi_2 = .5$
- $r = 11.11\%$

Determine ψ_3.

Solution:

$$\frac{1}{1+r} = \sum_{i=1}^{3} \psi_i \Rightarrow .9 = .2 + .5 + \psi_3 \Rightarrow \boxed{\psi_3 = .2}$$

Martingales and Submartingales

Suppose at time t one has information summarized by I_t. A random variable X_t that for all $s > 0$ satisfies the equality:

$$E^P[X_{t+s}|I_t] = X_t$$

is called a *martingale* with respect to the probability P.

If instead, we have for all $s > 0$:

$$E^P[X_{t+s}|I_t] \geq X_t$$

then X_t is called a *submartingale* with respect to the probability P.

Notes on Martingales

- This is important because discounted asset prices are martingales in the risk-neutral world only
- Must define X_{t+s} as the discounted future asset price $\frac{S_{t+s}}{(1+r)^s}$

Binomial Tree Pricing

- Project underlying out through binomial tree
- Determine the value at expiration using $C_T = \max(0, S_T - K)$
- Work backwards in the binomial tree for each time step until the root of the tree is reached
  - $S_t = \frac{1}{1+r}[Q_{up}(S_t + \sigma\sqrt{\Delta}) + Q_{down}(S_t - \sigma\sqrt{\Delta})]$
  - $C_t = \frac{1}{1+r}[Q_{up}C^{up}_{t+\Delta} + Q_{down}C^{down}_{t+\Delta}]$.

Dividends and Foreign Currencies

- Assume that dividends d are paid as a percentage of $S_{t+\Delta}$
- $\begin{bmatrix} 1 \\ S_t \\ C_t \end{bmatrix} = \begin{bmatrix} B^u_{t+\Delta} & B^d_{t+\Delta} \\ S^u_{t+\Delta} + d_t S^u_{t+\Delta} & S^d_{t+\Delta} + d_t S^d_{t+\Delta} \\ C^u_{t+\Delta} & C^d_{t+\Delta} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$
- $S = \frac{1+d}{1+r}[S^u Q_{up} + S^d Q_{down}]$
- $C = \frac{1}{1+r}[C^u Q_{up} + C^d Q_{down}]$
- $E^Q[\frac{S_{t+\Delta}}{S}] = \frac{1+r\Delta}{1+d\Delta} \approx 1 + (r-d)\Delta$
- $E^Q[\frac{C_{t+\Delta}}{C}] = 1 + r\Delta$

- Under risk-neutral expectation, the underlying S grows below the risk free rate at $r - d$ while the call option expected return is r.
- This should make intuitive sense – the holder of the underlying will receive d in dividends to compensate for the lower growth rate. The holder of the call option does not receive the dividends, so the expected return is simply the risk free rate.
- Similar idea with foreign currencies – replace d with the foreign savings interest rate r^f.

Generalizations and Extensions Needed

- Continuous time, $t \in [0, \infty)$
- Possibly uncountable number of states of the world
- Continuous discounting, $e^{-r\Delta}$

Risk-Neutral Pricing Summary

- Obtain a model to track the dynamics of the underlying
- Calculate how the derivative asset price relates to the price of the underlying asset at expiration or other boundaries
- Obtain risk-adjusted probabilities
- Calculate expected payoffs of derivatives at expiration using these risk-adjusted probabilties
- Discount this expectation using the risk-free return

Appendix: Generalizations of the Arbitrage Theorem

Define a portfolio:

$$\theta = \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \\ \cdots \\ \theta_n(t) \end{bmatrix}$$

such that the portfolio value at time t is $S_t'\theta = \sum_{i=1}^{n} S_i(t)\theta_i$

Definition: θ is an arbitrage portfolio, or simply an arbitrage, if either one of the following conditions is satisfied:

1. $S'\theta \leq 0$ and $D'\theta > 0$ (costs nothing to purchase, guaranteed positive return)
2. $S'\theta < 0$ and $D'\theta \geq 0$ (negative cost, nonnegative return)

$S'\theta$ is the purchase price, $D'\theta$ gives the payoff.

MFD Chapter 10: Ito's Lemma

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Overview of This Reading

This is one of the most important readings on the syllabus! We will explore Ito's Lemma, one of the most fundamental tools in stochastic calculus. **I highly recommend watching the Ito's Lemma and Baby Ito's Lemma review videos as you work through this chapter.** Also keep in mind there is an entire drill problem set in the practice tab of the seminar devoted to Ito's Lemma, and I recommend you work these drill problems while going through this chapter.

Key topics for the exam include:

- State Ito's Lemma and apply Ito's Lemma to examples
- Understand the five Ito's Lemma examples we will see

Warning: If you are following along in the textbook, it has major typos and mistakes. In particular, the statement of Ito's Lemma in both differential and integral form in the source material are incorrect. These have been corrected in the TIA seminar. Please keep this in mind!

Deterministic Review of Derivatives

- $F(S_t, t)$ is a function depending on two variables
  - t is simply a time variable
  - S_t is the stock price, which is dependent on t
- Partial Derivatives
  - $F_s = \frac{\partial F(S_t,t)}{\partial S_t}$
  - $F_t = \frac{\partial F(S_t,t)}{\partial t}$
  - Idea of partial derivatives is the same in both deterministic and stochastic calculus
  - Used for delta hedging
- "Total Derivatives" or "Differentials"
  - $dF_t = F_s dS_t + F_t dt = \frac{\partial F(S_t,t)}{\partial S_t} dS_t + \frac{\partial F(S_t,t)}{\partial t} dt$
- Chain Rule
  - $\frac{dF(S_t,t)}{dt} = F_s \frac{dS_t}{dt} + F_t$
  - Has complications to apply directly, since, for example, $\frac{dF(S_t,t)}{dt}$ cannot be defined for a Wiener process since in general F will not be smooth.
- Ito's Lemma will give us a tool for understanding *stochastic differentials*

Derivative Type	Example
Partial Derivatives	$F_s = \frac{\partial F(S_t,t)}{\partial S_t}$
Total Derivatives	$dF_t = \frac{\partial F(S_t,t)}{\partial S_t} dS_t + \frac{\partial F(S_t,t)}{\partial t} dt$
Chain Rule	$\frac{dF(S_t,t)}{dt} = F_s \frac{dS_t}{dt} + F_t$

Dropping Terms in the Stochastic Differential

- The following convention is used:

  Given a function $g(\Delta W_k, h)$ dependent on the increments of the Wiener process W_t, and on the time increment, consider the ratio:

  $$\frac{g(\Delta W_k, h)}{h}$$

  If this ratio vanishes in the mean square sense as $h \to 0$, then we consider $g(\Delta W_k, h)$ as negligible in small intervals. Otherwise, $g(\Delta W_k, h)$ is nonnegligible.

- Time components will be the same as deterministic derivatives. We need the first derivative, but can drop the higher order terms

- The more interesting components to check are the derivatives with respect to ΔW_k

- Informally, we can think about which terms are in the stochastic differential by the following baseball analogy. Assume that dt is like getting 2 strikes. Since $(dW_t)^2 = dt$, think of dW_t as 1 strike. Three strikes and you are out (of the Taylor expansion):

$\cdot$	**1**	dt	$(dt)^2$
1	$1 =$ 0 strikes	$dt =$ 2 strikes	$(dt)^2 =$ 4 strikes
(dW_t)	$dW_t =$ 1 strike	$dW_t \cdot dt =$ 3 strikes	$dW_t \cdot (dt)^2 =$ 5 strikes
$(dW_t)^2$	$(dW_t)^2 =$ 2 strikes	$(dW_t)^2 dt =$ 4 strikes	$(dW_t)^2 (dt)^2 =$ 6 strikes

Ito's Lemma

- Suppose that:

  1a. $F(S_t, t)$ is a twice-differentiable function of t and of the random process S_t

  2a. $dS_t = a_t dt + \sigma_t dW_t$

  3a. a_t and σ_t are well-behaved drift and diffusion parameters

- Then:

  1b. $dF_t = \frac{\partial F}{\partial S_t} dS_t + \frac{\partial F}{\partial t} dt + \frac{1}{2} \sigma_t^2 \frac{\partial^2 F}{\partial S_t^2} dt$

  2b. $dF_t = [a_t \frac{\partial F}{\partial S_t} + \frac{\partial F}{\partial t} + \frac{1}{2} \sigma_t^2 \frac{\partial^2 F}{\partial S_t^2}] dt + \frac{\partial F}{\partial S_t} \sigma_t dW_t$

Ito's Lemma Typos in Book

- The book drops the a_t and the squared part on σ_t^2. These are terrible typos! (2b) may be the most important statement in the entire book.
- There is also an extra dt written in the book, which is corrected in the above formula.

Proof Sketch

- While I will not (and the book does not) rigorously prove it, the idea is that we drop the higher order terms.
- The only remaining terms are dS_t, dt, and $(dS_t)^2 = (a_t dt + \sigma_t dW_t)^2 = \sigma_t^2 dt$
- Therefore, $(dS_t)^2 = \sigma_t^2 dt$
- Dropping higher order terms, $dF_t = \frac{\partial F}{\partial S_t} dS_t + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} (dS_t)^2$
- Combining these gives (1b): $dF_t = \frac{\partial F}{\partial S_t} dS_t + \frac{\partial F}{\partial t} dt + \frac{1}{2} \sigma_t^2 \frac{\partial^2 F}{\partial S_t^2} dt$
- Going from (1b) to (2b) is simple. Just substitute (2a) into (1b):

  $dF_t = \frac{\partial F}{\partial S_t}(a_t dt + \sigma_t dW_t) + \frac{\partial F}{\partial t} dt + \frac{1}{2} \sigma_t^2 \frac{\partial^2 F}{\partial S_t^2} dt$

  $\Rightarrow dF_t = [a_t \frac{\partial F}{\partial S_t} + \frac{\partial F}{\partial t} + \frac{1}{2} \sigma_t^2 \frac{\partial^2 F}{\partial S_t^2}] dt + \frac{\partial F}{\partial S_t} \sigma_t dW_t$

Ito's Lemma Notes

- Ito's Lemma is the stochastic extension of the deterministic chain rule
- Take a look at the similarities and difference between the deterministic total derivative and (1b)
  - Deterministic: $dF_t = \frac{\partial F}{\partial S_t} dS_t + \frac{\partial F}{\partial t} dt$
  - Stochastic: $dF_t = \frac{\partial F}{\partial S_t} dS_t + \frac{\partial F}{\partial t} dt + \frac{1}{2} \sigma_t^2 \frac{\partial^2 F}{\partial S_t^2} dt$
- Ito's formula takes the underlying asset price behavior (1b) as a given and then gives an SDE for financial derivatives
- Stochastic equality, as usual, is assumed in the mean-square sense. That is:

$$A = B \Rightarrow \text{Var}(A - B) \to 0 \text{ as } n \to \infty$$

Uses of Ito's Lemma

1. Provides a tool for obtaining stochastic differentials for functions of random variables
2. Helps with evaluating certain integrals as well. The 4 steps for this process are (for examples, see Q3-4 below):
   - Guess a function $F(W_t, t)$
   - Use Ito's Lemma to obtain the SDE for $F(W_t, t)$
   - Apply the integral operator to both sides of the SDE and simplify known integrals.
   - Rearrange to solve for the desired integral

Examples

Note: These five examples are extremely important. Past exam questions often leverage the results from these examples. You should be very comfortable and familiar with these five examples, as they are fundamental examples for understanding stochastic calculus and Ito's Lemma. For more information on substituions where $a = 0$ and $\sigma = 1$, make sure to check out the Baby Ito's Lemma Review Video.

1Q: Let W_t be a standard Wiener process with $F(W_t, t) = W_t^2$. Find the stochastic differential for dF_t

1A:
- $dS_t = dW_t \rightarrow a = 0$ and $\sigma = 1$
- Using Ito's Lemma, $dF_t = [\underbrace{a_t \frac{\partial F}{\partial S_t}}_{0} + \underbrace{\frac{\partial F}{\partial t}}_{0} + \underbrace{\frac{1}{2}\sigma_t^2 \frac{\partial^2 F}{\partial S_t^2}}_{1}]dt + \frac{\partial F}{\partial S_t}\sigma_t dW_t = dt + 2W_t dW_t$
- $dF_t = dt + 2W_t dW_t$

2Q: Let W_t be a standard Wiener process with $F(W_t, t) = 3 + t + e^{W_t}$. Find the stochastic differential for dF_t

2A:
- $dS_t = dW_t \rightarrow a = 0$ and $\sigma = 1$
- Using Ito's Lemma, $dF_t = [\underbrace{a_t \frac{\partial F}{\partial S_t}}_{0} + \underbrace{\frac{\partial F}{\partial t}}_{1} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 F}{\partial S_t^2}]dt + \frac{\partial F}{\partial S_t}\sigma_t dW_t$

  $= [1 + \frac{1}{2}\frac{\partial^2 F}{\partial S_t^2}]dt + \frac{\partial F}{\partial S_t} dW_t = [1 + \frac{1}{2}e^{W_t}]dt + e^{W_t} dW_t$
- $dF_t = [1 + \frac{1}{2}e^{W_t}]dt + e^{W_t} dW_t$

3Q: Evaluate the integral $\int_0^t s dW_s$

3A: It might not be obvious but we can do this if we apply Ito's Lemma with $F(W_t, t) = tW_t$. Then $a = 0$ and $\sigma = 1$.

- $dF_t = [\underbrace{a_t \frac{\partial F}{\partial S_t}}_{0} + \underbrace{\frac{\partial F}{\partial t}}_{W_t} + \underbrace{\frac{1}{2}\sigma_t^2 \frac{\partial^2 F}{\partial S_t^2}}_{0}]dt + \underbrace{\frac{\partial F}{\partial S_t}\sigma_t}_{t} dW_t$
- $dF_t = W_t dt + t dW_t$
- Thus, $\int_0^t dF_s = \int_0^t d[sW_s] = \int_0^t W_s ds + \int_0^t s dW_s$
- Also, $\int_0^t d[sW_s] = tW_t$
- Therefore, $\displaystyle\int_0^t s dW_s = tW_t - \int_0^t W_s ds$
- This does not feel very satisfying. The integral is still a function of another (simpler) integral

4Q: Evaluate the integral $\displaystyle\int_0^t W_s dW_s$

4A: The choice of F is not immediately obvious, but assume $F(W_t, t) = \frac{1}{2}W_t^2$.

- Using Ito's Lemma, $dF_t = [\underbrace{a_t \frac{\partial F}{\partial S_t}}_{0} + \underbrace{\frac{\partial F}{\partial t}}_{0} + \underbrace{.5\sigma_t^2 \frac{\partial^2 F}{\partial S_t^2}}_{.5}]dt + \underbrace{\frac{\partial F}{\partial S_t}\sigma_t}_{W_t} dW_t$
- $dF_t = .5dt + W_t dW_t$
- $\int_0^t dF_s = \int_0^t .5ds + \int_0^t W_s dW_s$
- $.5W_t^2 = .5t + \int_0^t W_s dW_s$
- $\displaystyle\int_0^t W_s dW_s = .5W_t^2 - .5t$

5Q: Calculate the expected value $E(F(t))$ where $F(t) = e^{\sigma W_t}$.

5A: This expectation is important and closely tied to geometric brownian motion and the lognormal distribution. It is possible to calculate this through Ito's Lemma. This example is actually in Chapter 11, and we will use our answer from this example in Chapter 11.

A few more quick notes before delving into the solutions. I'm putting in two solutions. The first is the approach taken in the textbook (page 187 in the derivation of equation 11.50). The first solution solves by using Ito's Lemma. However, I did want to also show a second solution which is much faster and leverages the property of a lognormal random variable.

Solution #1: Ito's Lemma Approach

- By now, hopefully it is clear that when we are using $W_t = S_t$
- Ito's Formula simplifies to the following:
- $dF_t = [\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S_t^2}]dt + \frac{\partial F}{\partial S_t} dW_t$
- $dF = [\underbrace{\frac{\partial F}{\partial t}}_{0} + \underbrace{\frac{1}{2}\frac{\partial^2 F}{\partial S_t^2}}_{.5\sigma^2 F}]dt + \underbrace{\frac{\partial F}{\partial S_t}}_{\sigma F} dW_t$
- $dF = .5\sigma^2 F dt + \sigma F dW_t$

- $\int_0^t dF_s = \int_0^t .5\sigma^2 F_s ds + \int_0^t \sigma F_s dW_s$
- Note that $F_0 = e^{\sigma W_0} = e^0 = 1 \Rightarrow E(F_0) = 1$
- Combining the equations above and taking expectations yields the equation below. Note that the last integral on the right hand side cancels out because of the martingale property (i.e. the expectation of an Ito integral equals 0)
- $E(F_t) - E(F_0) = E(\int_0^t .5\sigma^2 F_s ds) + E(\int_0^t \sigma F_s dW_s)$
  $= \int_0^t .5\sigma^2 E(F_s) ds = .5\sigma^2 \int_0^t E(F_s) ds$
- Now we have $E(F_t) - 1 = .5\sigma^2 \int_0^t E(F_s) ds$
- Note that now the integral is with respect to ds, so we can revert to deterministic calculus
- Set $x_t = E(F_t)$ which gives $x_t - 1 = .5\sigma^2 \int_0^t x_s ds$
- Differentiate both sides[1] of the above to get $dx_t = .5\sigma^2 x_t dt$
- Rearranging gives: $\frac{dx_t}{x_t} = .5\sigma^2 dt \Rightarrow \ln(X_t) = .5\sigma^2 t$
- The solution to this is $x_t = e^{.5\sigma^2 t}$
- Therefore, $x_t = E(F_t) = \boxed{E(e^{\sigma W_t}) = e^{.5\sigma^2 t}}$

Solution #2: Lognormal Approach

- Note that $\sigma W_t \sim N(0, \sigma^2 t)$
- Thus, this is simply the expectation of a lognormal random variable!
  - $E(L) = e^{\mu + .5\sigma^2}$
- Therefore, $\boxed{E(e^{\sigma W_t}) = e^{.5\sigma^2 t}}$

Integral Form of Ito's Lemma

- Stochastic differentials are simply a shorthand for Ito integrals over small intervals
- Using the fact that $\int_0^t dF_u = F(S_t, t) - F(S_0, 0)$ and integrating (2b) from Ito's Lemma gives the Integral Form of Ito's Lemma below
- $\int_0^t \sigma_u F_s dW_u = [F(S_t, t) - F(S_0, 0)] - \int_0^t [a_u F_s + F_u + .5F_{ss}\sigma_u^2] du$
- The equation in the book, like in Ito's Lemma, is not quite right. It has been fixed in the above equation

[1] By applying the fundamental theorem of calculus

Two Extensions to Ito's Lemma

1. Multivariate settings
2. Jumps

#1. Multivariate Settings

Consider a 2-dimensional setting with stock prices $S_1(t)$ and $S_2(t)$:

$$\begin{bmatrix} dS_1(t) \\ dS_2(t) \end{bmatrix} = \begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix} dt + \begin{bmatrix} \sigma_{11}(t) & \sigma_{12}(t) \\ \sigma_{21}(t) & \sigma_{22}(t) \end{bmatrix} \begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix}$$

- Assume that $W_1(t)$ and $W_2(t)$ are independent
- Notice that there are two noise processes, and there in general will be correlated noise feeding into $dS_1(t)$ and $dS_2(t)$ unless $\sigma_{12}(t) = \sigma_{21}(t) = 0$
- F is now in a slightly more complicated form, $F(S_1(t), S_2(t), t)$
- We can solve for dF_t using the multivariate Ito's Lemma
- $dF_t = F_t dt + F_{S_1} dS_1 + F_{S_2} dS_2 + \frac{1}{2}[F_{S_1S_1} dS_1^2 + F_{S_2S_2} dS_2^2 + 2F_{S_1S_2} dS_1 dS_2]$
- The above is not quite the equivalent of Ito's Lemma, since we would like to substitute in for the dS terms
- Using similar reasoning, one must go through and see which terms drop out of the stochastic differential
- It can be shown that we can plug in the follow dS terms:
  - $dS_1(t)^2 = [\sigma_{11}^2(t) + \sigma_{12}^2(t)]dt$
  - $dS_2(t)^2 = [\sigma_{21}^2(t) + \sigma_{22}^2(t)]dt$
  - $dS_1(t)dS_2(t) = [\sigma_{11}(t)\sigma_{21}(t) + \sigma_{12}(t)\sigma_{22}(t)]dt$
- Example: If you wish to use a two-factor interest rate model using a short rate and long rate, extending to the multivariate setting of Ito's Lemma will be required.
- Example: Wealth
  - $N_i(t)$: units of ith asset
  - $P_i(t)$: price of ith asset
  - n total assets
  - $N_i(t)$ and $P_i(t)$ are assumed to be continuous time processes, potentially a function of the same random shocks
  - Let $Y(t)$ be the total value of the investment at time t

- Then we have that $Y(t) = \sum_{i=1}^{n} N_i(t)P_i(t)$
- We can calculate the increments in wealth as time passes (note the last term!):

$$dY(t) = \sum_{i=1}^{n} N_i(t)dP_i(t) + \sum_{i=1}^{n} dN_i(t)P_i(t) + \sum_{i=1}^{n} dN_i(t)dP_i(t)$$

#2. Ito's Formula with Jumps

- Consider the following SDE for S:

$$dS_t = a_t dt + \sigma_t dW_t + dJ_t$$

- Note that we are adding on an extra term here, dJ_t, to account for the jump component.
- This is a second innovation term, and since it is an innovation term, it must have mean 0:
  - $E(\Delta J_t) = 0$
- Between jumps, J_t remains constant. The jumps have sizes a_i that occur at rate λ_t. Once a jump occurs, its size a_i is determined with probability p_i
- There are then two sources of randomness for the jump process: the frequency and the size of jumps
- Let N_t be the sum of all jumps up to time t
- $\Delta J_t = \underbrace{\Delta N_t}_{\text{Actual Jumps}} - \underbrace{[\lambda_t h(\sum_{i=1}^{k} a_i p_i)]}_{\text{Expected Jumps}}$
- $\underbrace{a_t}_{\text{Total Drift}} = \underbrace{\alpha_t}_{\text{Continuous Drift}} + \underbrace{\lambda_t(\sum_{i=1}^{k} a_i p_i)}_{\text{Jump Drift}}$
- Using Ito's Lemma, the SDE for F is

  $$dF(S_t, t) = [F_t + \lambda_t \sum_{i=1}^{k} [(F(S_t + a_i, t) - F(S_t, t))p_i] + .5F_{ss}\sigma^2]dt + F_s dS_t + dJ_F$$

  where we have dJ_F given by:

  - $dJ_F = [F(S_t, t) - F(S_t^-, t)] - \lambda_t \cdot [\sum_{i=1}^{k} (F(S_t + a_i, t) - F(S_t, t))p_i]dt$
  - $S_t^- = \lim_{s \to t} S_s, s < t$

Conclusion

- Ito's Lemma is a key tool in stochastic calculus
- Ito's Lemma helps to determine stochastic differentials for financial derivatives, given movements in the underlying asset
    - To see this, note that the stock price process behavior (2a) is an input to Ito's Lemma, and the output is the stochastic behavior of a function of the stock price process
- In the statement of Ito's Lemma, equality should be interpreted in the mean square sense of stochastic equivalence